Notes on the Connection between the Ideal and Two-Fluid Plasma Electric Field with Application to Bennett-Shumlak Vortices

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May 25, 2026

Consider Gauss' Law for the electric field due to the presence of charge carriers in vacuum,

$$\nabla \cdot \vec{E} = \frac{\rho_c}{\epsilon_0} = e(Zn_i - n_e)\frac{1}{\epsilon_0} \tag{1}$$

along with a purely axial flow, i.e., a Z-pinch,

$$\vec{u} = u_z \hat{z} \tag{2}$$

$$\vec{J} = e(Zn_i - n_e)u_z \hat{z} \tag{3}$$

$$\vec{B} = B_\theta \hat{\theta} = \frac{\mu_0}{r}\int_0^r r' J_z(r') dr' \hat{\theta} \tag{4}$$

What is required so that the two-fluid electric field from Gauss' Law is the same as the ideal theory where Ohm's Law gives,

$$\vec{E} = \vec{u} \times \vec{B} \tag{5}$$

$$= -u_z B_\theta \hat{r} \tag{6}$$

? We have from Equation (1) that,

$$E_r = \frac{e}{r\epsilon_0}\int_0^r (Zn_i - n_e) r' dr' \tag{7}$$

Equating this to Equation (6) yields,

$$\frac{e}{r\epsilon_0}\int_0^r (Zn_i - n_e) r' dr' = -u_z \frac{\mu_0}{r}\int_0^r e(Zn_i - n_e) r' u_z dr' \tag{8}$$

If we treat u_z as non-uniform then we face the challenge of guessing what the correct pairing with plasma number density is to yield equality in the above. Pursuit along this line is possibly aided by treating the plasma current density as being that of a Bennett-Shumlak vortex, for example the cubic, pureflow form,

$$\vec{J} = J_z(r)\hat{z} = -en_0 u_{z,0}\frac{r^2}{(r + C_{B,T})^2}\hat{z} \tag{9}$$

as this can provide an analytic means by which the plasma density and flow velocity are coupled. However, this line still faces the aforementioned challenge so instead we will consider what happens if the flow is uniform.

In this case, when the flow is uniform, we pull it out of the integral on the RHS and find that the same integral remains on both sides of the expression so that Equation (8) becomes,

$$-\frac{u_z^2}{c^2} = 1 \tag{10}$$

where,

$$c^2 = \frac{1}{\epsilon_0 \mu_0} \tag{11}$$

was used. The negative sign can be neglected for the purposes of studying the amplitude of the electric field, as it only expresses a notion of direction. Then, the constraint which makes the amplitude of this two-fluid electric field equal to the ideal case is that the flow speed be equal to the speed of light.

The direction of this electric field however would point outwards instead of inwards against the plasma pressure gradient. This tracks with the loss of confinement that unsheared Z-pinches experience on ultrafast timescales, and suggests that we need to consider a shear form to the plasma flow instead. The simplest shear form we can describe would involve a uniform density to avoid dealing with problems of diffusion or continuity so we would have then,

$$-\frac{u_z}{c^2}\int_0^r r' u_z(r')dr' = \frac{r^2}{2} \tag{12}$$

Equation (12) requires specification of the axial flow. The need for a uniform density here reduces the number of possible flow patterns that we can study while still remaining connected to a shear-flow stabilized Bennett-Shumlak equilibrium. The simplest pattern to study is the cubic, pureflow vortex,

$$u_z(r) = u_{z,0}\frac{r^2}{(r + C_{B,T})^2} \tag{13}$$

Inserting this into Equation (12) we have,

$$\begin{aligned}
\frac{r^2}{2} &= -u_z \frac{1}{c^2}\int_0^r r' u_z(r')dr' \qquad (14)\\
&= -\frac{u_z}{c^2}\int_0^r \frac{r'^3}{(r' + C_{B,T})^2}dr' \qquad (15)\\
&= -\frac{u_z}{c^2}\frac{f(r, C_{B,T})}{2r(r + C_{B,T})} \qquad (16)
\end{aligned}$$

where,

$$f(r, C_{B,T}) = r^3 - 3r^2 C_{B,T} \tag{17}$$

$$- 6rC_{B,T}^2 \left(1 + \ln\left(\frac{r}{r + C_{B,T}}\right)\right) \tag{18}$$

$$- 6C_{B,T}^3 \ln\left(\frac{r}{r + C_{B,T}}\right) \tag{19}$$

the connection between electric fields in this regime then resides on finding roots to the rational polynonimal,

$$r(r + C_{B,T})^3 c^2 + f(r, C_{B,T}) = 0 \tag{20}$$

Because none of these terms involve powers greater than quartic it is suggested that these roots can be solved for analytically using Ruffini's construction.