Supplementary Information for Bennett Vortices

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Weak Form

Under the exchange of the Bennett nonlinearity from density to flow we have,

$$u_z(r) = \frac{u_{z,0}}{(1+\xi^2 r^2)^2} \tag{1}$$

where,

$$\xi^2 = bn_0 = \frac{C_B n_0}{T(r)} \tag{2}$$

with,

$$C_B = \frac{\mu_0 e^2 u_{z,0}^2}{16 k_B} \tag{3}$$

The Shumlak criterion for the shear-flow stabilization of a Z-pinch in vacuum reads,

$$\frac{\mathrm{d}u_z}{\mathrm{d}r} > 0.1 k V_A = \frac{\pi}{5L}\frac{B}{\sqrt{\rho\mu_0}} \tag{4}$$

and the magnetic Reynold's number,

$$R_m = \sigma\mu u L \tag{5}$$

For a fixed, and finite, Z-pinch plasma the length of the pinch can be taken as infinite in order to study the consequences of the ideal theory of magnetohydrodynamics for this profile. Then the Shumlak criterion reads,

$$\frac{\mathrm{d}u_z}{\mathrm{d}r} > 0 \tag{6}$$

provided that the magnetic field does not go to infinity, or that the density does not become trivial. What then is the condition on the temperature profile in order to satisfy this criteria and obtain a shear-flow stabilized Z-pinch?

$$\begin{aligned}
\frac{\mathrm{d}u_z}{\mathrm{d}r} &= \frac{\mathrm{d}}{\mathrm{d}r}\left[u_{z,0}\frac{T(r)^2}{(T(r)+C_B n_0 r^2)^2}\right] && (7)\\
&= u_{z,0}\left[2TT'(T+C_B n_0 r^2)^{-2} - 2T^2(T+C_B n_0 r^2)^{-3}(T'+2C_B n_0 r)\right] && (8)\\
&= u_{z,0}\left[\frac{2TT'(T+C_B n_0 r^2)}{(T+C_B n_0 r^2)^3} - \frac{2T^2(T'+2C_B n_0 r)}{(T+C_B n_0 r^2)^3}\right] && (9)\\
&= u_{z,0}\left[\frac{2TT'C_B n_0 r^2 - 4T^2 C_B n_0 r}{(T+C_B n_0 r^2)^3}\right] && (10)\\
&= u_{z,0}\left[2TC_B n_0 r\frac{rT'-2T}{(T+C_B n_0 r^2)^3}\right] && (11)\\
&> 0 && (12)
\end{aligned}$$

Restricting to $r > 0$, the denominator is strictly positive, and doe snot affect the sign of the inequality. Even if it is zero it does not matter as the RHS will remain unchanged, but there is the matter that this would result in the fallacious $0/0 = 1$ if $r = 0$, and $T(0) = 0$. However, we work around this by turning our attention away from the axis as the plasma state is specified entirely there by our choices, so that we do not have to worry about such a problem in the rest of the pinch where we wish to understand the necessary structure of $T(r)$ for our construction.

$$\therefore \left[2TC_B n_0 r(rT' - 2T)\right] > 0 \tag{13}$$

$$\rightarrow (rT' - 2T) > 0 \tag{14}$$

for the same reason that we can throw away the earlier denominator, because we are restricting our attention away from the origin where an absolute zero temperature would pose an issue to our machinations. From here we simply treat this conditional as an integrable differential equation, move infinitesimals around the way physicists do, and arrive at,

$$\frac{T'}{T} > \frac{2}{r} \tag{15}$$

$$\rightarrow \frac{dT}{T} > \frac{2}{r}dr \tag{16}$$

$$\therefore \ln(T) > 2\ln(r) \tag{17}$$

$$\therefore e^{\ln(T)} > e^{2\ln(r)} \tag{18}$$

$$\implies T(r) > r^2 \tag{19}$$

is sufficient. However, this is the "weak form", and it is called as such because it requires the existence of an integral, and expresses a shear of zero in the pure-flow profile when the temperature is taken to be greater than a parabola, while also being parabolic in form,

$$T(r) = C_T^{(2)} r^2 = \frac{T_p}{r_p^2} r^2 \tag{20}$$

$$\therefore \frac{\mathrm{d}u_z}{\mathrm{d}r} = \frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{u_{z,0}}{(1 + \frac{C_B n_0}{C_T^{(2)}} \frac{r^2}{r^2})^2}\right] = 0 \tag{21}$$

Of course, it is also highly advised to consider,

$$T(r) = T_0 \pm C_T^{(2,0)} r^2 \tag{22}$$

as a more realistic weak form whose consequences should be investigated since the inverse parabolic form is what is seen in experimental temperature profiles. However, that is outside the scope of this supplement as the results presented in the article do not involve this case explicitly.

Strong Form

The result of the weak form invites a consideration of what happens when a power-law is adopted for the temperature profile,

$$T(r) = C_T^{(n)} r^n = \frac{T_p}{r_p^n} r^n \tag{23}$$

and this is termed the "strong form" because it does not hinge on the existence of an integral, and it expresses a non-trivial shear. Skipping back to the end of the Shumlak criterion for this new case, we have,

$$(rT' - 2T) > 0 \tag{24}$$
$$\therefore rT' > 2T \tag{25}$$
$$\rightarrow rC_T^{(n)} n r^{n-1} > 2C_T^{(n)} r^n \tag{26}$$
$$\implies n > 2 \tag{27}$$

Cubic Pure-flow Profile

Since the $n = 2$ case is a weak satisfaction expressing a trivial shear unless there is a core temperature added into the expression, $n = 3$ is the first case to check for whether there is actually a soluble shear-flow stabilized equilibrium which can be curtailed from the base flow profile,

$$u_z^{(2,3)}(r) = \frac{u_{z,0}}{(1 + \frac{C_B n_0}{C_T^{(3)}} \frac{r^2}{r^3})^2} \tag{28}$$
$$= \frac{u_{z,0}}{(1 + \frac{C_{B,T}^{(3)}}{r})^2} \tag{29}$$
$$= u_{z,0} \frac{r^2}{(r + C_{B,T}^{(3)})^2} \tag{30}$$

where it was convenient to lump all the plasma properties and constants into,

$$C_{B,T}^{(3)} = C_B n_0 \frac{r_p^3}{T_p} \tag{31}$$

which is the origin of the parenthetical "3" superscript, i.e., the cubic temperature profile supporting this structure, with the "2" being present for consistency with the article notation where it is present to indicate this is the $\chi = 2$ member of an entire family predicated on this fundamental act of exchange whose existence is being shown in this supplement.

The shear of this profile is,

$$\frac{\mathrm{d}u_z^{(2,3)}}{\mathrm{d}r} = \frac{\mathrm{d}}{\mathrm{d}r}\left[u_{z,0} \frac{r^2}{(r + C_{B,T}^{(3)})^2}\right] \tag{32}$$
$$= u_{z,0}\left[2r(r + C_{B,T})^{-2} - 2r^2(r + C_{B,T})^{-3}\right] \tag{33}$$
$$= u_{z,0}\left[\frac{2r(r + C_{B,T}) - 2r^2}{(r + C_{B,T})^3}\right] \tag{34}$$
$$= u_{z,0} \frac{2rC_{B,T}}{(r + C_{B,T})^3} \tag{35}$$

where the parenthetical superscript was dropped for brevity in the $C_{B,T}$ shear-layer parameter. The shear at the origin is well-defined, and can be evaluated by inspection to dispel concerns about our projection of focus away from the pinch axis when deriving this profile,

$$\left.\frac{\mathrm{d}u_z}{\mathrm{d}r}\right|_{r=0} = 0 \tag{36}$$

Points of extremal shear occur when the derivative of the shear is itself 0,

$$u_z'' = 2C_{B,T}u_{z,0}\frac{\mathrm{d}}{\mathrm{d}r}\left[r(r+C_{B,T})^{-3}\right] \tag{37}$$

$$= 2C_{B,T}u_{z,0}\left[(r+C_{B,T})^{-3} - 3r(r+C_{B,T})^{-4})\right] \tag{38}$$

$$= 2C_{B,T}u_{z,0}\left[\frac{C_{B,T}-2r}{(r+C_{B,T})^4}\right] \tag{39}$$

Setting the above equal to zero yields,

$$r_{max} = \frac{C_{B,T}}{2} \tag{40}$$

The shear at this location is,

$$u'_{z,max} = \frac{2\frac{C_{B,T}}{2}C_{B,T}u_{z,0}}{(\frac{C_{B,T}}{2}+C_{B,T})^3} \tag{41}$$

$$= \frac{C_{B,T}^2 u_{z,0}}{(\frac{3}{2}C_{B,T})^3} \tag{42}$$

$$= \frac{8}{27}\frac{u_{z,0}}{C_{B,T}} \tag{43}$$

With these basic properties established, our next step is to evaluate the equilibrium as it proceeds entirely from this pure-flow profile. There is no *a priori* guarantee that this profile will integrate to produce a magnetic field, or that this magnetic field will further integrate to produce a salient plasma pressure. Obtaining the plasma pressure via integration of the momentum equation is preferred to the ideal gas law closure because Noether's Theorem, which states that conservation laws, like that of a momentum balance, are a consequence of symmetries, like that of the translational invariance of the laws of physics, is the stronger support compared to an *ad hoc* closure relation based on a rarefied gas picture.

The magnetic field is obtained from Ampere's Law in ideal magnetohydrodynamics, and for a cylindrical Z-pinch equilibrium this field only has an azimuthal component,

$$B_\theta(r) = \frac{\mu_0}{r}\int_0^r r' J_z(r')dr' \tag{44}$$

$$= \frac{-en_0 u_{z,0}\mu_0}{r}\int_0^r \frac{r'^3}{(r'+C_{B,T})^2}dr' \tag{45}$$

This integral can be evaluated analytically with a u-substitution,

$$u = r' + C_{B,T} \tag{46}$$

$$\therefore B_\theta(r) = -\frac{en_0\mu_0 u_{z,0}}{r}\int_{u=C_{B,T}}^{u=r+C_{B,T}} \frac{(u-C_{B,T})^3}{u^2}du \tag{47}$$

$$= -\frac{en_0\mu_0 u_{z,0}}{r}\int_{u=C_{B,T}}^{u=r+C_{B,T}} \frac{u^3-3u^2C_{B,T}+3uC_{B,T}^2-C_{B,T}^3}{u^2}du \tag{48}$$

$$= -\frac{en_0\mu_0 u_{z,0}}{r}\int_{u=C_{B,T}}^{u=r+C_{B,T}} u-3C_{B,T}+3\frac{C_{B,T}^2}{u}-\frac{C_{B,T}^3}{u^2}du \tag{49}$$

$$= -\frac{en_0\mu_0 u_{z,0}}{r}\left[\frac{u^2}{2}-3C_{B,T}u+3C_{B,T}^2\ln(u)+\frac{C_{B,T}^3}{u}\right|_{u=C_{B,T}}^{u=r+C_{B,T}} \tag{50}$$

$$= -\frac{en_0\mu_0 u_{z,0}}{r}\left[\frac{(r+C_{B,T})^2}{2}-3C_{B,T}(r+C_{B,T})+3C_{B,T}^2\ln(r+C_{B,T})+\frac{C_{B,T}^3}{r+C_{B,T}}\right. \tag{51}$$

$$\left.-\frac{C_{B,T}^2}{2}+3C_{B,T}^2-3C_{B,T}^2\ln(C_{B,T})-C_{B,T}^2\right] \tag{52}$$

or with Wolfram Mathematica to yield the form presented in the article. Evaluating via u-sub is equivalent after some algebra,

$$B_\theta(r) = -en_0u_{z,0}\mu_0\left(\frac{r^2-3rC_{B,T}-6C_{B,T}^2}{2(r+C_{B,T})}+\frac{3C_{B,T}^2\ln\left(\frac{r+C_{B,T}}{C_{B,T}}\right)}{r}\right) \tag{53}$$

$$= -en_0u_{z,0}\mu_0\frac{r^3-3r^2C_{B,T}-6rC_{B,T}^2+6(r+C_{B,T})C_{B,T}^2\ln\left(\frac{r+C_{B,T}}{C_{B,T}}\right)}{2r(r+C_{B,T})} \tag{54}$$

$$= \frac{-\mu_0en_0u_{z,0}}{2r(r+C_{B,T})}f(r,C_{B,T}) \tag{55}$$

which will require the application of the log property,

$$\ln(a/b) = \ln(a)-\ln(b) = -\ln(b/a) \tag{56}$$

The plasma pressure kernel is determined by this magnetic field, and the structure of the B-field is crucial to the validity of this equilibrium in the context of the Shumlak criterion, so we will integrate this plasma pressure kernel first to see if a full equilibrium is produced,

$$p(r) = p_0 - \int_0^r J_zB_\theta dr' \tag{57}$$

$$p(r_p) = 0 \implies p_0 = \int_0^{r_p} J_zB_\theta dr \tag{58}$$

$$\therefore p(r) = p_0 - \int_0^r -en_0u_{z,0}\frac{r'^2}{(r'+C_{B,T})^2}\left(-\frac{\mu_0en_0u_{z,0}}{2r'(r'+C_{B,T})}f(r',C_{B,T})\right)dr' \tag{59}$$

$$= p_0 - \frac{\mu_0}{2}(en_0u_{z,0})^2\int_0^r \frac{r'}{(r'+C_{B,T})^3}f(r',C_{B,T})dr' \tag{60}$$

$$= p_0 - \frac{\mu_0}{2}(en_0u_{z,0})^2\int_0^r \frac{r'}{(r'+C_{B,T})^3}(f_1+f_2+f_3+f_4)dr' \tag{61}$$

Despite its appearance the above integral is tractable as it splits into four pieces,

$$\int_0^r \frac{r'}{(r'+C_{B,T})^3}(f_1+f_2+f_3+f_4)dr' = P_1+P_2+P_3+P_4 \tag{62}$$

$$P_1 = \int_0^r \frac{r'^4}{(r' + C_{B,T})^3} dr' \tag{63}$$

$$P_2 = -3C_{B,T} \int_0^r \frac{r'^3}{(r' + C_{B,T})^3} dr' \tag{64}$$

$$P_3 = -6C_{B,T}^2 \int_0^r \frac{r'^2}{(r' + C_{B,T})^3} \left(1 + \ln\left(\frac{C_{B,T}}{r' + C_{B,T}}\right)\right) dr' \tag{65}$$

$$P_4 = -6C_{B,T}^3 \int_0^r \frac{r'}{(r' + C_{B,T})^3} \ln\left(\frac{C_{B,T}}{r' + C_{B,T}}\right) dr' \tag{66}$$

which are either rational functions amenable to u-substitution or the product of a rational function and a logarithmic term that can be integrated by parts after appropriate substitution.

Bulk Profiles & Mixed Vorticity

The addition of a uniform background flow can be used to extend the cubic pure-flow profile to more realistic situations where the pinch axis will not be in stagnation. There is a deep consequence of this fact, namely that there is a *mixed* nature at the heart of the fundamental character of this result which can be seen from considering the inclusion of a bulk profile in the electron plasma current density,

$$\vec{J} = J_z \hat{z} = -en(r)(u_0 \pm u_{z,0} \frac{r^2}{(r + C_{B,T})^2}) \tag{67}$$

If the $n(r)$ in the above is selected appropriately so that an analytic plasma current density remains then the combination of flow and density is naturally tractable. The ability to distribute the Bennett nonlinearity reflects this notion, as exponents can be attached to the density and flow to describe how much of the original nonlinearity is exchanged,

$$n^{(\nu)}(r) = n_0 \frac{T^\nu}{(T + C_B n_0 r^2)^\nu} \tag{68}$$

$$u_z^{(\chi)}(r) = u_{z,0} \frac{T^\chi}{(T + C_B n_0 r^2)^\chi} \tag{69}$$

where the constraint that,

$$\chi + \nu = 2 \tag{70}$$

exists so that we arrive at the basic form of the analytic current density we obtained,

$$\vec{J} = -en_0 u_{z,0} \frac{T(r)^2}{(T(r) + C_B n_0 r^2)^2} \hat{z} \tag{71}$$

Taking this a step further, we know that a cubic temperature profile in the above leads to an analytic Z-pinch equilibrium based on the plasma current density,

$$\vec{J} = -en_0 u_{z,0} \frac{r^2}{(r + C_{B,T}^{(3)})^2} \hat{z} \tag{72}$$

Consequently, any combination of flow and density which produces this current density will also result in the same analytic Z-pinch equilibrium as before,

$$\vec{J} = -en(r) u_z(r) \hat{z} \tag{73}$$

Another point to note is that the inclusion of a uniform background current density is also tractable,

$$\vec{J} = \vec{J}_0 - en_0 u_{z,0} \frac{r^2}{(r + C^{(3)}_{B,T})^2} \hat{z} \tag{74}$$

This can be seen trivially because the uniform background integrates in the aforementioned manner to produce a tractable magnetic field. One note to make is that in practice, the analytic magnetic field of a cubic vortex contains logarithmic terms which introduce numerical oscillations into the computation of any equilibrium properties. It is arguably better then, in practice, to integrate the magnetic field numerically when these numerical oscillations would otherwise be present in the analytic form.

To close the section, let us consider a plasma current density representing the cubic, pure-flow ($\chi = 2$), Bennett vortex but which is described by a flow that includes a uniform background,

$$u_z(r) = u_0 + u_{z,0} \frac{r^2}{(r + C_{B,T})^2} \tag{75}$$

so that the plasma current density is,

$$J_z(r) = -en_0(u_0 + u_{z,0} \frac{r^2}{(r + C_{B,T})^2}) \tag{76}$$

It is worth noting that the above form could be also constructed out of a non-uniform density and a cubic pure-flow profile and the results would be the same, albeit with a complicated non-uniformity describing the density. The exchange of sign from positive to negative has no influence on the tractability of the equations, and will only introduce a change of sign into the relevant parts of the result so the positive is taken here without loss of generality.

The magnetic field clearly integrates as the uniform background flow becomes a uniform background plasma current when the density is considered uniform, yielding,

$$B_\theta(r) = -en_0\mu_0 \left(\frac{u_0}{2} r + u_{z,0} \frac{f(r)}{2r(r + C_{B,T})} \right) \tag{77}$$

Incredibly, the pressure integrates in this case as well, being done here with the Wolfram Mathematica CAS due to the complexity of the expressions,

$$p(r) - p_0 = -\mu_0 e^2 n_0^2 \Big(P_1 + P_2 + P_3 + P_4 \Big) \tag{78}$$

$$P_1 = \int_0^r \frac{u_0^2}{2} r' dr' \tag{79}$$

$$P_2 = \int_0^r u_0 u_{z,0} \frac{f(r', C_{B,T})}{2r'(r' + C_{B,T})} dr' \tag{80}$$

$$P_3 = \int_0^r u_0 u_{z,0} \frac{r'^3}{2(r' + C_{B,T})^2} dr' \tag{81}$$

$$P_4 = \int_0^r \frac{u_{z,0}^2}{2} \frac{r' f(r', C_{B,T})}{(r' + C_{B,T})^3} dr' \tag{82}$$

P_1 is trivial, and P_3 is a re-scaled version of the previous cubic integral giving the functional form for the magnetic field of a pure-flow vortex. P_2, and P_4 are substantially more complicated.

Interestingly, P_2 introduces a complex pressure, and a Jonquiere function which is typically seen only in quantum statistics rather than classical plasma physics,

$$\begin{aligned} P_2 = \frac{1}{4} u_0 u_{z0} \Bigg\{ & r^2 - 8rC_{B,T} \\ & - 2C_{B,T}^2 \Bigg[\pi^2 - 6\,\mathrm{arctanh}\left(\frac{r}{r+2C_{B,T}}\right) \\ & + 5\ln\left(1+\frac{r}{C_{B,T}}\right) + 6\ln\left(r\left(1+\frac{r}{C_{B,T}}\right)\right)\ln(C_{B,T}) \\ & + 6i\pi\ln\left(\frac{C_{B,T}}{r+C_{B,T}}\right) - 6\ln(r)\ln(r+C_{B,T}) \\ & - 6\,\mathrm{Li}_2\left(1+\frac{r}{C_{B,T}}\right)\Bigg]\Bigg\} \end{aligned} \tag{83}$$

where $Li_2(1+\frac{r}{C_{B,T}})$ is the second-order Jonquiere function (polylogarithm) typically seen in quantum statistics when integrating Fermi-Dirac, or Bose-Einstein, distributions. Its emergence here in a classical, ideal, MHD context is suggestive of a deeper structural connection.

Flow Boundary Conditions

Before continuing on past this section to evaluate the shear-flow stabilized character with the Shumlak criterion, Equation (4), we must take a minute to discuss the boundary conditions of these cases. In the cubic, pure-flow ($\chi = 2$) case we have,

$$u_{edge} = u_z(r_p) = u_{z,0}\frac{r_p^2}{(r_p + C_{B,T})^2} \tag{84}$$

The primary wrinkle in the above is the existence of an additional quadratic factor of $u_{z,0}$ inside of the shear layer placement $C_{B,T}$. If this value is sufficiently small, meaning,

$$C_{B,T} << r_p \tag{85}$$

then the flow constant root becomes degenerate,

$$u_{edge} \simeq u_{z,0} \tag{86}$$

Otherwise the boundary condition must be translated into an algebraic system,

$$u_{edge}(r_p + C_{B,T})^2 - u_{z,0}r_p^2 = 0 \tag{87}$$
$$\therefore u_{edge}(r_p^2 + 2r_pC_{B,T} + C_{B,T}^2) - u_{z,0}r_p^2 = 0 \tag{88}$$
$$\implies Au_{edge}u_{z,0}^4 + 2Bu_{edge}u_{z,0}^2 - r_p^2u_{z,0} + u_{edge}r_p^2 = 0 \tag{89}$$

where

$$A = \frac{\mu_0^2 e^4 n_0^2 r_p^6}{(16k_BT_p)^2} \tag{90}$$

$$B = \frac{\mu_0 n_0 e^2 r_p^4}{16k_BT_p} \tag{91}$$

that can technically be solved as all quartic equations can to give four different solutions for $u_{z,0}$. Nothing requires these roots to be real, and a pattern of complex roots shows up in the results presented in the article. The existence of complex roots relates to the structure of these quartic solutions which are based on closed form expression obtained in the 1800s by the Italian mathematician Ruffini[?]. An in-depth exploration of this topic is beyond the scope of this supplement.

In the bulk case we have,

$$u_{edge} = u_0 \pm u_{z,0}\frac{r_p^2}{(r_p + C_{B,T})^2} \tag{92}$$

$$\therefore u_{edge} - u_0 = \pm u_{z,0}\frac{r_p^2}{(r_p + C_{B,T})^2} \tag{93}$$

so that the structure of the algebraic system remains largely unchanged except for the substitution implied by the above. Of course, a subtlety is introduced by this, for if $u_{edge} = u_0$, then only trivial solutions to the flow roots exist. Evidently, within this equilibrium family it is inadmissible for there to be a symmetry between the edge and core flows, because such a uniformity would cause the shear to collapse between the span of the column as a consequence of the flow stagnation.

Minimum Pinch Lengths

We must evaluate the Shumlak criterion, Equation (4) for these profiles to determine the minimum length required for the Z-pinch equilibrium to be shear-flow stabilized. This analysis will consider the limit as the plasma radius goes to zero so that the behavior of this property in an arbitrarily small space will be elucidated.

There are four primary forms to evaluate this criterion for in this supplement, namely, the cubic, n-form, and their bulk forms. Specifically, we are evaluating the expression,

$$L > \frac{\pi}{5}(\rho\mu_0)^{-1/2}\left(\frac{\mathrm{d}u_z}{\mathrm{d}r}\right)^{-1} B_\theta \tag{94}$$

for its value as $\lim_{r\to 0}$ is taken. This expression can be simplified by considering a length that is normalized by the uniform values on the RHS, and this can include the mass density if we restrict our study to situations which admit the treatment of a uniform density. In principle, this means any plasma current density that describes a Bennett vortex without loss of generality because a vortex with a non-uniform density can be treated as isomorphic to the case of a uniform density and the flow patterns investigated in this article.

Then, we study

$$\lim_{r\to 0} L^* > \left(\frac{\mathrm{d}u_z}{\mathrm{d}r}\right)^{-1} B_\theta \tag{95}$$

$$L^* = L/\left(\frac{\pi}{5}(\rho\mu_0)^{-1/2}\right) \tag{96}$$

The influence of a bulk flow will only impact the magnetic field form so we begin with the non-bulk cases, and first amongst them the cubic case,

$$\lim_{r\to 0} L_3^* > -\frac{en_0\mu_0}{u_{z,0}C_{B,T}}\lim_{r\to 0}\frac{(r + C_{B,T})^3}{r}\frac{f(r)}{2r(r + C_{B,T})} \tag{97}$$

$$> -\frac{en_0\mu_0}{2u_{z,0}C_{B,T}}\lim_{r\to 0}\frac{(r + C_{B,T})^2}{r^2}f(r) \tag{98}$$

Two applications of L'Hopitals rule are required to lift the singularity in the denominator. The limit we wish to evaluate then becomes,

$$\lim_{r\to 0} 2f + 4(r + C_{B,T})f' + (r + C_{B,T})^2 f'' = 2f(0) + 4C_{B,T}f'(0) + C_{B,T}^2 f''(0) \tag{99}$$

The derivatives that need to be evaluated are,

$$f'(r) = -6C_{B,T}r + 3r^2 + \frac{6C_{B,T}^3}{r + C_{B,T}} + \frac{6C_{B,T}^2 r}{r + C_{B,T}} - 6C_{B,T}^2\left(1 + \ln\left(\frac{C_{B,T}}{r + C_{B,T}}\right)\right) \tag{100}$$

$$f''(r) = -6C_{B,T} + 6r - \frac{6C_{B,T}^3}{(r + C_{B,T})^2} - \frac{6C_{B,T}^2 r}{(r + C_{B,T})^2} + \frac{12C_{B,T}^2}{r + C_{B,T}} \tag{101}$$

At the origin they become, alongside $f(r)$,

$$f(0) = 0 - 0 - 6C_{B,T}^3 \ln(1) - 6\Big(1 + \ln(1)\Big) * 0 * C_{B,T}^2 = 0 \tag{102}$$

$$f'(0) = 0 + 0 + 6C_{B,T}^2 + 0 - 6C_{B,T}^2\Big(1 + \ln(1)\Big) = 0 \tag{103}$$

$$f''(0) = -6C_{B,T} + 0 - 6C_{B,T} + 0 + 12C_{B,T} = 0 \tag{104}$$

evidently, we find,

$$\lim_{r\to 0} L_3^* > 0 \tag{105}$$

This limit suggests a divergence analogous to an ultraviolet-type pathology, as the only non-trivial singularity this result can suffer from in the remaining coupling is when $T_p \to \infty$ since this causes the shear structure to collapse, shown here by an explosion of the minimum length required for this equilibrium to assume a shear-flow stabilized state in an arbitrarily small space towards ∞.

The addition of a bulk flow does not change the structure of the shear but it does change the form of the magnetic field on the RHS of the Shumlak criterion,

$$\lim_{r\to 0} L_{3,0}^* > -\frac{en_0\mu_0}{u_{z,0}C_{B,T}} \lim_{r\to 0} \frac{(r + C_{B,T})^3}{r}\left(\frac{u_0}{2}r \pm \frac{f(r)}{2r(r + C_{B,T})}\right) \tag{106}$$

$$> -\frac{en_0\mu_0}{2u_{z,0}} u_0 C_{B,T}^2 \tag{107}$$

This is negative-valued, and when the pinch radius goes to zero it does as well because of the $\sim r_p^6$ scaling of the RHS.

The last cases to consider involve when the power-law of the temperature profile is defined by an arbitrary integer, n,

$$T(r) = C_T^{(n)} r^n = \frac{T_p}{r_p^n} r^n \tag{108}$$

The magnetic field in this case is,

$$\begin{aligned} B_{\theta;2,n} = -\frac{en_0\mu_0 u_{z,0}}{(n-2)^2 r}\Bigg(&\frac{(n-2)r^{2n}}{C_{B,T}^{(n)}(C_{B,T}^{(n)}r^2 + r^n)} \\ &- (-\frac{1}{C_{B,T}^{(n)}})^{-\frac{2}{n-2}} n\beta_I(-\frac{r^{n-2}}{C_{B,T}^{(n)}}, 2 + \frac{2}{n-2}, 0)\Bigg) \end{aligned}$$

where,

$$\beta_I(z;a,b) = \int_0^z u^{a-1}(1-u)^{b-1}du \tag{109}$$

is the incomplete beta function. A plasma pressure was not found when an attempt was made, but this is enough to evaluate the minimum pinch length necesssary for an arbitrarily small Z-pinch equilibrium of this kind to form a shear-flow stabilized state,

The shear of a pure-flow n-vortex is,

$$u_z(r) = u_{z,0}\frac{T^2}{(T+C_B n_0 r^2)^2} = u_{z,0}\frac{1}{(1+C_{B,T}^{(n)}r^{2-n})^2} \tag{110}$$

$$\left.\frac{\mathrm{d}u_z}{\mathrm{d}r}\right|_{2,n} = -2u_{z,0}(2-n)\frac{r^{1-n}}{(1+C_{B,T}^{(n)}r^{2-n})^3} \tag{111}$$

so we have,

$$L_n(0) > \lim_{r\to 0} -\frac{\pi}{10}(\rho\mu_0)^{-1/2}\frac{(1+C_{B,T}^{(n)}r^{2-n})^3}{(2-n)u_{z,0}}r^{n-1}B_{\theta;2,n}(r) \tag{112}$$

expanding out the magnetic field obtains,

$$L_n(0) > -\frac{\pi}{10}(\rho\mu_0)^{-1/2}\frac{en_0\mu_0}{C_{B,T}^{(n)}}\lim_{r\to 0} r^{n-2}\frac{(1+C_{B,T}^{(n)}r^{2-n})^3}{(2-n)(n-2)^2}\Bigg[\frac{(n-2)r^{2n}}{C_{B,T}^{(n)}r^2(C_{B,T}^{(n)}+r^{n-2})} \tag{113}$$

$$-(-\frac{1}{C_{B,T}^{(n)}})^{-\frac{2}{n-2}}n\beta_I(-\frac{r^{n-2}}{C_{B,T}^{(n)}},2+\frac{2}{n-2},0)\Bigg] \tag{114}$$

In the given limit the beta function will go to zero because the size of the space its kernel is being integrated across does. Importantly, this kernel remains well-defined here because $a > 1$, so there is no objection to be made on the grounds of the kernel losing its regularity. The remaining portion then becomes,

$$L_n^*(0) > \lim_{r\to 0} r^{n-2}(1+C_{B,T}^{(n)}r^{2-n})^3\frac{(n-2)r^{2n}}{C_{B,T}^{(n)}(C_{B,T}^{(n)}r^2+r^n)} \tag{115}$$

$$> \frac{n-2}{C_{B,T}^{(n)}}\lim_{r\to 0}\frac{(C_{B,T}^{(n)}r^2+r^n)^2}{r^2} \tag{116}$$

$$> 0 \tag{117}$$

which can be evaluated by applying L'Hopitals Rule twice, and where,

$$L_n^* = \frac{L_n}{C_{L,2,n}} \tag{118}$$

$$C_{L,2,n} = -\frac{\pi}{10}(\rho\mu_0)^{-1/2}\frac{en_0\mu_0}{C_{B,T}^{(n)}}\frac{1}{(2-n)(n-2)^2} \tag{119}$$

Evidently, every member of this part of the family of shear-flow stabilized Bennett vortices can form in this crucially stable state for arbitrarily small spaces. Note that the above also implies that the weak form of $n = 2$ will also have an arbitrarily small requirement for pinch length.

The only remaining case to consider from the original four discussed is when a bulk flow is added to that of an n-vortex. The shear remains unchanged, and as in the previous bulk case we have already shown that the pure-flow contribution will be nothing. That leaves us with,

$$L_{n,0}(0) > -\frac{e\pi n_0\mu_0}{5}(\rho\mu_0)^{-1/2}\lim_{r\to 0}\left(\frac{\mathrm{d}u_z}{\mathrm{d}r}\right)^{-1}\left(\frac{u_0}{2}r\right) \tag{120}$$

which boils down to,

$$\tilde{L}_{n,0}(0) > \lim_{r\to 0}\frac{(r^n + C_{B,T}^{(n)}r^2)^3}{r^{2n}} \tag{121}$$

This limit is indeterminate for general n, and shear layer. However, as can be seen previously from the cubic case this limit will evaluate for certain values of n, e.g., $n = 3$. In this specific case, it can also be seen that an arbitrarily small pinch length is sufficient for such an equilibrium to be shear-flow stabilized when the pinch radius is arbitrarily small as this will kill the quadratic term in the above and leave a single factor of r^n on the RHS.

A trivial shear layer also corresponds to the ultraviolet limit, and not just an infinitesimally vanishing pinch radius, so that is another aspect to consider because it is suggestive of the possibility that this equilibrium could possibly explain how the aftermath of the Big Bang stabilized into a quark-gluon plasma that then cooled and eventually condensed into ordinary matter. Letting the temperature run to infinity admits the possibility that an arbitrarily small pattern of this kind of flow could establish itself with a thermal lifetime and persist to interact with its fellows if there were a plasma state setting the stage for these events. What is interesting to close this section with as an observation, is that the same is not true for the cubic case since there the ultraviolet limit will cause the length to become indeterminate so that some relaxation process would first be necessary for the universe to cool off before these lower-order vortices could form.

Thermal Structure

The classic Bennett pinch treats a uniform plasma temperature and therefore matters of heat flux, thermal confinement time, and thermal energy go trivially unanswered. By itself a cubic Bennett vortex represents a steady-state heat flux distribution that can be treated as producing a net volumetric heating through its divergence,

$$\nabla\cdot\vec{q} = Q \tag{122}$$

A Braginskii type closure[**?**] will study the dressed thermal transport properties in greater detail than is required at a first pass to understand the basic thermal structure of the cubic vortex and how it responds to unsteady perturbations. For this first pass we consider a uniform perpendicular conductivity,

$$\vec{q} = -\kappa_\perp\nabla T \tag{123}$$

The cross-field, Righi-Leduc term,

$$\vec{q}_\wedge = -\kappa_\wedge\hat{b}\times\nabla T = \kappa_\wedge\frac{\mathrm{d}T}{\mathrm{d}r}\hat{z} \tag{124}$$

shows that an axial flux of heat will develop, but this heat will remain confined to the axis if the system is axially symmetric, and its transport properties uniform, because then there will be no divergence of the heat.

With a uniform thermal conductivity we have in steady state,

$$-\kappa_\perp \nabla^2 T = Q \tag{125}$$

The steady volumetric heating provided by an n-vortex,

$$T_n(r) = \frac{T_p}{r_p^n} r^n = C_T^{(n)} r^n \tag{126}$$

can be evaluated to give,

$$Q_n(r) = -\frac{\kappa_\perp}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}T_n}{\mathrm{d}r}\right) \tag{127}$$

$$= -\frac{\kappa_\perp}{r}\left(C_T^{(n)} n r^{n-1} + C_T^{(n)} n(n-1) r^{n-1}\right) \tag{128}$$

$$= -\kappa_\perp \frac{T_p}{r_p^n} r^{n-2}\left(n + n(n-1)\right) \tag{129}$$

$$= -\kappa_\perp \frac{T_p}{r_p^n} n^2 r^{n-2} \tag{130}$$

Integrating over the entire plasma volume gives the thermal power of the vortex,

$$-\int_{V_p} \kappa_\perp \nabla^2 T_n dV = \int_{V_p} Q_n dV = S_n \tag{131}$$

For an n-vortex we have,

$$S_n = -\kappa_\perp 2\pi L \int_0^{r_p} \frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}T_n}{\mathrm{d}r}\right) dr \tag{132}$$

$$= -2\pi L \kappa_\perp T_p \frac{1}{r_p^n}\int_0^{r_p} n^2 r^{n-1} dr \tag{133}$$

$$= -2\pi L \kappa_\perp T_p n^2 \frac{1}{r_p^n}\frac{r_p^n}{n} \tag{134}$$

$$= -2\pi n L \kappa_\perp T_p \tag{135}$$

after the integration is performed. Amongst the strong-form solutions the total thermal power is then minimized for the cubic temperature case when $n = 3$ and the system is axially symmetric with a uniform thermal conductivity. Parabolic temperatures have less thermal power, but they are only weakly shear-flow stabilized and have mathematically zero shear.

The energy confinement time[**?**] is defined according to the pressure and heat flux at the pinch boundary,

$$\frac{3}{2}\frac{p_0}{\tau_E} = \frac{1}{V_p}\int_{A_L} \vec{q}\cdot d\vec{A}_L \tag{136}$$

For a cubic vortex it can be found to be,

$$\tau_E = \frac{1}{12}\frac{p_0}{\kappa_\perp(r_p) T_p} r_p^2 \tag{137}$$

where the plasma pressure at the core of the pinch can be solved for by taking the pressure at the boundary to be equal to the vacuum pressure. In the pure-flow, $\chi = 2$ case with a pure vacuum this yields,

$$p_0 = \frac{\mu_0(en_0u_{z,0})^2}{2}\int_0^{r_p}\frac{r}{(r+C_{B,T})^3}f(r,C_{B,T})dr \tag{138}$$

$$= \frac{\mu_0(en_0u_{z,0})^2}{2(r_p+C_{B,T})^2}\Bigg(r_p(2C_{B,T}+r_p)(r_p^2-12C_{B,T}r_p-15C_{B,T}^2) \tag{139}$$

$$+6C_{B,T}^2(C_{B,T}+r_p)\ln\left(\frac{C_{B,T}}{r_p+C_{B,T}}\right)((C_{B,T}+r_p)\ln\left(\frac{C_{B,T}}{r_p+C_{B,T})}\right) \tag{140}$$

$$-3r_p-5C_{B,T})\Bigg)$$

at the plasma edge the (perpendicular) thermal conductivity is[?],

$$\kappa_\perp = 4.7\frac{n_0k_BT_e}{m_e\omega_{ce}^2\tau_e} \tag{141}$$

$$\tag{142}$$

and the maximum magnetic field, which occurs at the edge of the plasma, is,

$$B_\theta(r_p) = B_{max} \tag{143}$$

$$= \frac{\mu_0en_0u_{z,0}}{2r_p(r_p+C_{B,T})}f(r_p,C_{B,T}) \tag{144}$$

Because of the linear nature of the source and sink terms in the power balance of the plasma, any collection of sources and sinks, e.g., fusion power, Ohmic heating, Bremsstrahlung losses, etc. can be represented by an n-vortex whose temperature defines the natural scale of the net conductive flux of heat at the plasma edge. Equating the net volumetric power output of these sources and sinks with the net flux of conductive thermal power from the plasma at the pinch radius, we have,

$$\int_{A_L}\vec{q}\cdot d\vec{A}_L = V_p\sum_k S_k \tag{145}$$

$$\rightarrow 2\pi Lr_p\kappa_\perp(r_p)\frac{T_v}{r_p^n}nr_p^{n-1} = \pi r_p^2L\sum_k S_k \tag{146}$$

$$\therefore T_v = \frac{r_p^2}{2n\kappa_\perp(r_p)}\sum_k S_k \tag{147}$$

where the volumetric power density of the kth source or sink is S_k, and the temperature gradient at the plasma edge is,

$$\left.\frac{\mathrm{d}T}{\mathrm{d}r}\right|_{r=r_p} = \frac{T_v}{r_p^n}nr_p^{n-1} = \frac{T_v}{r_p}n \tag{148}$$

Plasma Filament Formation

The plasma being a circular disk of uniform density, the impact of its gravitational forces measured along the positive half-chord of the pinch cross-section can be calculated analytically by treating the gravitational forces along the cylinder of mass in the classical sense, i.e., as if all the mass in a given shell were concentrated at the center of the plane, and a central-force was pulling on the test mass, m, of plasma at some point, r. The mass M of the shell is given by,

$$M = 2\pi L \int_0^r r' \rho dr' \tag{149}$$
$$= L\pi r^2 m n_0 \tag{150}$$

and the gravitational force on the test mass, m, in the lab frame,

$$\vec{F}_G = -\frac{GM(r)m}{r^2}\hat{r} \tag{151}$$

The ratio of the gravitational force to the electromagnetic forces felt by the test plasma mass in the lab frame can be calculated for the case $T(r) = C_T r^3$ since the entire equilibrium of a Bennett Vortex can be for this situation. The radial electric field felt by the test plasma mass in its rest frame can necessarily be obtained as well, from the ideal version of Ohm's Law,

$$\vec{E} = -\vec{u} \times \vec{B} = E_r \hat{r} = \frac{r}{(r + C_{B,T})^3} \mu_0 e n_0 u_{z,0}^2 f(r, C_{B,T}) \tag{152}$$

If we use this electric field in a naive attempt to calculate the ratio of these two forces we will find that the electromagnetic force is considered to have no impact on the dynamics of the plasma mass, and this is by definition. Instead, to restore the impact of the electromagnetic force we must boost to a lab frame travelling at a velocity, $\vec{u}_{lab}$,

$$\vec{u}_{lab} \times \vec{B} = -2\vec{u} \times \vec{B} \tag{153}$$
$$\therefore \vec{u}_{lab} = -2\vec{u} \tag{154}$$

Doing so amounts to finding a frame of reference where the electric force adds constructively with the magnetic force instead of cancelling it out fully, as is done in the rest frame of the plasma. Critically, the system remains axisymmetric from this new frame of reference,

$$z_{lab} = z - 2u_z(r)t \tag{155}$$
$$\therefore \frac{\partial q(r)}{\partial z_{lab}} = \frac{\partial z}{\partial z_{lab}} \frac{\partial q}{\partial z} = \frac{\partial q}{\partial z} = 0 \tag{156}$$

and furthermore, this frame of reference is also inertial as the plasma flow is steady.

In principle any scaling factor could be coupled to the boost for performing this calculation, subsequently changing the value of the electromagnetic forces seen by the lab observer, but doing so would cause the amplitude of the lab electric field to grow or shrink from its value in the rest frame of the plasma. Furthermore, applying too large a scaling factor, or consider too large an edge flow speed, and the relativistic effects on the observed mass from the lab frame will need to be addressed.

When measured in this frame, the electric field experienced by the plasma mass becomes,

$$\vec{E}_{lab} = \vec{u} \times \vec{B} \tag{157}$$

and the electromagnetic forces in the lab frame become,

$$\vec{F}_{L,lab} = e(\vec{E}_{lab} + \vec{u} \times \vec{B}) \tag{158}$$
$$= 2e(\vec{u} \times \vec{B}) \tag{159}$$
$$= F_{L,lab,r}\hat{r} \tag{160}$$
$$= -2eu_z B_\theta \hat{r} \tag{161}$$

Defining the ratio of the gravitational force to this lab electromagnetic force as,

$$\eta_{GL} = \frac{\int_V |\vec{F}_G(r)| dV}{\int_V |\vec{F}_{L,lab}(r)| dV} = \frac{I_G}{I_{EM}} \tag{162}$$

we have, first, the electromagnetic integral,

$$\begin{aligned} I_{EM} &= 2\pi L e^2 u_{z,0}^2 n_0 \mu_0 \int_0^r \frac{f(r', C_{B,T})}{(1+\xi^2 r'^2)^4 (r' + C_{B,T})} dr' \\ &= 2\pi L e^2 u_{z,0}^2 n_0 \mu_0 (L_1 + L_2 + L_3 + L_4) \end{aligned} \tag{163}$$

where,

$$L_1 = \int_0^r \frac{f_1(r', C_{B,T})}{(1+\xi^2 r'^2)^4 (r' + C_{B,T})} dr' \tag{164}$$

$$L_2 = \int_0^r \frac{f_2(r', C_{B,T})}{(1+\xi^2 r'^2)^4 (r' + C_{B,T})} dr' \tag{165}$$

$$L_3 = \int_0^r \frac{f_3(r', C_{B,T})}{(1+\xi^2 r'^2)^4 (r' + C_{B,T})} dr' \tag{166}$$

$$L_4 = \int_0^r \frac{f_4(r', C_{B,T})}{(1+\xi^2 r'^2)^4 (r' + C_{B,T})} dr' \tag{167}$$

These integrals, Equations (164) - (167), are solved using the Wolfram Mathematica CAS. The same techniques can be used for this as were used for evaluating the pressure integrals, u-substitution and integration by parts for when the rational functions are coupled to a logarithmic term. What is important is their leading order when added together,

$$I_{EM} \sim \frac{r^7}{(r + C_{B,T})^4} \sim r^3 \tag{168}$$

and the leading order of the gravitational integral,

$$I_G = 2\pi L G m \int_0^r \frac{1}{r'} M(r') dr' \tag{169}$$

$$= 2\pi^2 L^2 G m^2 n_0 \int_0^r r' dr' \tag{170}$$

$$= (\pi^2 L^2 G m^2 n_0) r^2 \tag{171}$$

Together we find that the ratio, Equation (162) is then given by,

$$\eta_{GL} \sim \frac{L^2 r^2}{L r^3} \sim \frac{L}{r} \tag{172}$$

for this particular, non-relativistic, case where $T = C_T r^3$.

For a relativistic flow in a flat spacetime the pinch length, but not radius because there is no radial flow treated in this frame, contracts in the flow frame to a new length, L',

$$L' = \frac{1}{\gamma(v)} L = L\sqrt{1 - \frac{v^2}{c^2}} \tag{173}$$

In the ultra-relativistic limit, $v \approx c$, which is naturally entered for sufficiently high energy particles or when the current grows sufficiently large in a plasma whose electrons are carrying it, then this length goes to zero. Ultra-relativistic particles in this equilibrium then would not see a pinch at all as they simply "jump" to the end of it once borne. This suggests a mechanism by which the structure can form, namely, high-energy electrons which propagate on the fastest timescale in a plasma. If fast electrons were to assume the equilibrium, then the rest of the plasma will have no choice but to move collectively with them for the proper duration of the structure's thermal lifetime.

The collapse of the pinch length in the fast electron frame suggests that a flat spacetime cannot support a continuous axial structure of this kind, indicating a preference for localized growth of filaments as the current density is amplified in a very narrow region by the shear structure of this equilibrium. When such amplification occurs, which can happen freely across scales ranging from small laboratory plasmas to very large astrophysical systems depending on the characteristics of the shear layer, this provides a unifying mechanism for the appearance of filaments.

Where this is challenged is if the length of the structure in the slow frame is very long as then accounting for the strict difference between the local speed of light and fast electron speed will give a finite pinch length in the fast frame. This also highlights the importance of accounting for the gravitational forces instead of neglecting them as otherwise it would be unclear what impact this length has on the overall impact a test plasma mass experiences. What meets these challenges is both the minimal energy of this Z-pinch equilibrium, and its shear-flow stabilized condition, as these force the growth of the electron plasma current into channels which can be arbitrarily narrow depending on the local state of the plasma.

Spherical Bennett Vorticity

It is valuable to study what form this flow takes in a spherical basis. Transforming from a cylindrical to spherical basis results in,

$$u_\rho = u_z(\rho, \phi)\cos(\phi) \tag{174}$$

$$u_\theta = 0 \tag{175}$$

$$u_\phi = -u_z(\rho, \phi)\sin(\phi) \tag{176}$$

where the relationship,

$$\rho^2 = r^2 + z^2 = r^2 + \rho^2\cos^2(\phi) \tag{177}$$

$$\therefore r^2 = \rho^2(1 - \cos^2(\phi)) \tag{178}$$

reveals that the axisymmetric cylindrical current density we have been studying suffers a broken symmetry in the spherical basis that is identified with the polar dependence necessary for this 2D flow to transform back into the 1D axial model. Interestingly, we see that the azimuthal symmetry is preserved. This is suggestive of the reason why the axial symmetry of the equilibrium is relaxable in the case of a spherical air plasma streamer head, because this axial symmetry is a phantom symmetry that breaks when the cylindrical basis is transformed into spherical.

The current density then stands as,

$$\vec{J} = J_\rho(\rho, \phi)\hat{\rho} + J_\phi(\rho, \phi)\hat{\phi} \tag{179}$$

which is the basic structure on which the magnetostatic field rests,

$$\boldsymbol{\nabla} \times \vec{B} = \mu_0 \vec{J} \tag{180}$$

$$\implies \frac{1}{\rho \sin\theta}\left(\frac{\partial(\sin\theta B_\phi)}{\partial\theta} - \frac{\partial B_\theta}{\partial\phi}\right)\hat{\rho} + \frac{1}{\rho}\left(\frac{\partial(\rho B_\theta)}{\partial\rho} - \frac{\partial B_\rho}{\partial\theta}\right)\hat{\phi} \tag{181}$$

$$\therefore \left(\boldsymbol{\nabla} \times \vec{B}\right)_\theta = 0 \implies \frac{1}{\rho\sin\theta}\frac{\partial B_\rho}{\partial\phi} = \frac{1}{\rho}\frac{\partial(\rho B_\phi)}{\partial\rho} \tag{182}$$

The above illustrates clearly that the azimuthal magnetic field splits into a radial and polar component with fully broken symmetries while the lack of azimuthal current in the magnetostatic case results in a coupling between the radial and polar magnetic field gradients. A full 3D treatment of the magnetic field is then required to resolve it from this perspective.

Resistive Bennett Vorticity

The incorporation of a finite resistivity is seamless into the power balance, as the electrical power of a uniform cylindrical ideal plasma is expressable as,

$$P = I_{encl}^2 R = I_{encl}^2 \frac{L}{\pi r_p^2}\eta \tag{183}$$

which is calculable for an arbitrary pure-flow vortex with pinch radius, r_p,

$$I_{encl} = \int_{A_p} \vec{J}\cdot d\vec{A}_p \tag{184}$$

and usage of the Spitzer resistivity[?],

$$\eta_{sp} = \frac{4\sqrt{2\pi}}{3}\frac{Z_{eff}e^2 m^{1/2}\ln(\Lambda_c)}{(4\pi\epsilon_0)^2(k_B T_e)^{3/2}} \tag{185}$$

where

$$Z_{eff} = \frac{\sum_j Z_j^2 n_j}{n_e} \tag{186}$$

and Z_j is the charge (ionization) state of the j-th ionic species.

The energy confinement time[?] is defined according to the pressure and heat flux at the pinch boundary,

$$\frac{3}{2}\frac{p_0}{\tau_E} = \frac{1}{V_p}\int_{A_L} \vec{q}\cdot d\vec{A}_L \tag{187}$$

For a cubic vortex it can be found to be,

$$\tau_E = \frac{1}{12}\frac{p_0}{\kappa(r_p)T_p}r_p^2 \tag{188}$$

where the plasma pressure at the core of the pinch can be solved for by taking the pressure at the boundary to be equal to the vacuum pressure. In the pure-flow, $\chi = 2$ case with a pure vacuum this yields,

$$p_0 = \frac{\mu_0(en_0u_{z,0})^2}{2}\int_0^{r_p}\frac{r}{(r+C_{B,T})^3}f(r,C_{B,T})dr \tag{189}$$

$$= \frac{\mu_0(en_0u_{z,0})^2}{2(r_p+C_{B,T})^2}\Bigg(r_p(2C_{B,T}+r_p)(r_p^2-12C_{B,T}r_p-15C_{B,T}^2) \tag{190}$$

$$+6C_{B,T}^2(C_{B,T}+r_p)\ln\left(\frac{C_{B,T}}{r_p+C_{B,T}}\right)((C_{B,T}+r_p)\ln\left(\frac{C_{B,T}}{r_p+C_{B,T})}\right) \tag{191}$$

$$-3r_p-5C_{B,T})\Bigg)$$

at the plasma edge the (perpendicular) thermal conductivity is[**?**],

$$\kappa_\perp = 4.7\frac{n_0k_BT_e}{m_e\Omega_e^2\tau_{ee}} \tag{192}$$

$$= 4.7\frac{\Lambda_C\sqrt{m_e}e^2 4\sqrt{2\pi}}{3}\frac{n_0^2}{B(r_p)^2\sqrt{k_BT_p}} \tag{193}$$

and the maximum magnetic field, which occurs at the edge of the plasma, is,

$$B_\theta(r_p) = B_{max} \tag{194}$$

$$= \frac{\mu_0en_0u_{z,0,}}{2r_p(r_p+C_{B,T})}f(r_p,C_{B,T}) \tag{195}$$

The Alfven time is another timescale which is of great importance to this kind of equilibrium as it describes the time it takes for magnetic energy, which can be of arbitrary shape, to be transported across a given lengthscale,

$$\tau_A = \frac{L^*}{V_A} \tag{196}$$

A plasma pinch has two natural lengthscales, the first being the pinch radius, r_p, and the second being the pinch length, L, as it is measured in a frame at rest with respect to the pinch. This suggests two Alfvenic timescales to consider,

$$\tau_{A;p} = \frac{r_p}{V_A} \tag{197}$$

and,

$$\tau_{A;L} = \frac{L}{V_A} \tag{198}$$

where the Alfven speed is of course,

$$V_A = \frac{B}{\sqrt{\rho\mu_0}} \tag{199}$$

We should consider the ratio of the confinement time against this time,

$$R_{EA} = \frac{\tau_E}{\tau_A} \tag{200}$$

Viscous (Incompressible) Bennett Vorticity

A Louiville equation shows how the impact of viscosity will influence the structure of the plasma dynamics in general, and with the full momentum equation being for a plasma species s interacting visco-collisionally with a background s',

$$m_s n_s \left(\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \nabla \vec{u} \right) = \vec{J} \times \vec{B} + \nabla \cdot \underline{\underline{\Pi}} + \sum_{s'} m_s n_s \nu_{ss'} (\vec{u}_{s'} - \vec{u}_s) \tag{201}$$

In the case of the shear-flow stabilized Z-pinch equilibrium, the flow is purely axial,

$$\vec{u} = u_z(r)\hat{z} \tag{202}$$

so for incompressible flow where the viscous stress constitutes a divergence with an isotropic viscosity,

$$(\nabla \cdot \underline{\underline{\Pi}})_z = \partial_j \Pi_{zj} = \mu_v \partial_j \partial_j u_z = \mu_v \frac{1}{r} \frac{\mathrm{d}(r \frac{\partial u_z}{\partial r})}{\mathrm{d}r} \tag{203}$$

then a first-pass Newtonian closure for a singly-ionized plasma presents a buildup of electric field on the z-axis, balanced by viscous momentum transfer in the absence of collisions,

$$0 = q_s n_0 E_z + (\nabla \cdot \underline{\underline{\Pi}})_z \tag{204}$$

$$= q_s n_0 E_z + \mu_v \frac{1}{r} \frac{\mathrm{d}(r \frac{\partial u_z}{\partial r})}{\mathrm{d}r} \tag{205}$$

Drifts

The very serious question of plasma drifts must also be considered as magnetization of the plasma brings on cyclotron emission, and leads to guiding-center drifts orthogonal to both the perpendicular electric, and magnetic fields of the plasma by introducing a finite ordering to the guiding-center motion based on the cyclotron frequency. Additionally, these equilibria have a magnetic null at the axis of the pinch so that there will be a region in which the magnetized cyclotron orbits transition to betatron orbits where the increasing nature of the magnetic field is what is important, rather than its amplitude. Meaning that, in general, a greater range of drifts must be considered than just those resulting from magnetic effects.

Drifts resulting from the isotropic pressure gradient, the non-uniform electromagnetic field of the vortex, and a possible gravitational field will be considered. No appreciable curvature drift will be present in the idealized equilibrium because there is no plasma flow considered to be occurring parallel to the magnetic field. Likewise there will be no polarization drift because the electric field is considered to be steady, although in the presence of a growth mode for a strong accelerating electric field this would not be the case. For axial drifts this leaves the grad-B drift,

$$\vec{v}_{\nabla B,j} = \pm \frac{1}{2} v_\perp \rho_L \frac{\vec{B} \times \vec{\nabla} B}{B^2} \tag{206}$$

$$= \pm \frac{1}{2} v_\perp^2 \frac{1}{\omega_{cj}} \frac{B_\theta \hat{\theta} \times B'_\theta(r)\hat{r}}{B^2} \tag{207}$$

$$= \mp \frac{1}{2} m_j v_\perp^2 \frac{1}{|q_j|} \frac{B'_\theta}{B^2} \hat{z} \tag{208}$$

$$= \mp \frac{K_{\perp,j}}{|q_j| B^2} B'_\theta \hat{z} \tag{209}$$

$$\tag{210}$$

the ExB drift with a non-uniform electric field,

$$\vec{v}_E = (1 + \frac{1}{4}\rho_L^2\nabla^2)\frac{\vec{E}\times\vec{B}}{B^2} \tag{211}$$
$$= (1 + \frac{\rho_L^2}{4}\nabla^2)\frac{(-u_z\hat{z}\times B_\theta\hat{\theta})\times B_\theta\hat{\theta}}{B^2} \tag{212}$$
$$= (1 + \frac{\rho_L^2}{4}\nabla^2)\frac{u_z B_\theta\hat{r}\times B_\theta\hat{\theta}}{B^2} \tag{213}$$
$$= (1 + \frac{\rho_L^2}{4}\nabla^2)u_z\hat{z} \tag{214}$$
$$\tag{215}$$

and the diamagnetic drift,

$$\vec{v}_D = \frac{1}{|q_j|n}\frac{-\frac{\mathrm{d}p}{\mathrm{d}r}\hat{r}\times B_\theta\hat{\theta}}{B^2} \tag{216}$$
$$= \frac{1}{|q_j|n}\frac{J_z B_\theta\hat{r}\times B_\theta\hat{\theta}}{B^2} \tag{217}$$
$$= -\frac{1}{ne}enu_z\hat{z} \tag{218}$$
$$= -u_z\hat{z} \tag{219}$$
$$\tag{220}$$

where,

$$K_\perp = \frac{1}{2}mv_\perp^2 \tag{221}$$
$$\rho_{L,j} = \frac{v_{\perp,j}}{\omega_{cj}} \tag{222}$$
$$\vec{E} = -\vec{u}\times\vec{B} \tag{223}$$

and $\rho_{L,j}$ is the Larmor radius of the jth plasma species. If this scale is small compared to the length-scale over which gradients in the plasma exist, then for a singly-ionized plasma with a current carried by the electrons we find the electric drift and diamagnetic drifts cancel in the present equilibrium, leaving the behavior in this regime dependent on the remaining drifts.

Waterfall Drifts

If gravity is considered then we must first, while recognizing our discussion remains firmly rooted in a flat spacetime, choose its setting. The most important aspect of this choice is its situation relative to the pinch axis, and the source of the field. If the field is parallel to the pinch axis, and uniform, so that the pinch is thought of as being located on the surface of a massive body which the gravitational field is due to, then we can say this configuration looks something like a waterfall, and the drift is,

$$\vec{v}_{wf} = \frac{m_j}{q_j}\frac{g_0\hat{z}\times B_\theta\hat{\theta}}{B^2} \tag{224}$$
$$= -\frac{g_0}{\omega_{cj}}\hat{r} \tag{225}$$

the weakly-magnetized interior of these vortices will result in a large such drift then near the core as indicated in the main article, and additionally because their magnetic field points in the opposite direction to $\hat{\theta}$ this drift will travel outwards rather than inwards as suggested above,

$$\vec{v}_{wf} = \frac{g_0}{\omega_{cj}}\hat{r} \tag{226}$$

Of course, this is the collisionless picture, and the weak magnetic field in the core of these shear-flow stabilized Z-pinch equilibria suggests that if the pinch axis of the discharge was aligned with the local gravitational field of a massive body on its surface, then significant radial drifts would be excited in these regions to transport matter radially in the plasma from the gravitational influence of the massive body. In a thermal plasma reactor where excited electrons are available to react with pollutant ions from a background discharge plasma, this process could be used to ionize, and transport industrial effluence out of the impurity gas for deposition on a material substrate as part of a sequestration pipeline.

The primary issue to address is how collisions influence this picture. In the absence of viscosity, or unsteady Eulerian effects, then concentrating on a plasma velocity that is made up purely of a radial "waterfall" component and the axial vortex flow presented in this article,

$$\vec{u} = u_{wf}(r)\hat{r} + u_z(r)\hat{z} \tag{227}$$

we have to contend with the existence of convective nonlinearities at the fluid scale. However, considering the force balance on a single particle where the effect of collisions are described with a drag term we have,

$$0 = q_s(\vec{E} + \vec{u}\times\vec{B}) + m_s g_0\hat{z} - m_s\nu\vec{v} \tag{228}$$

Taking the electric field to be zero in the above as a consequence of plasma screening we have,

$$0 = -q_s u_z B_\theta - m_s\nu u_r \tag{229}$$

$$0 = q_s u_r B_\theta + m_s g_0 - m_s\nu u_z \tag{230}$$

From here we can obtain,

$$u_r\left(1 + (\frac{m_s\nu}{q_s B_\theta})^2\right) = -\frac{g_0}{\omega_{cj}} \tag{231}$$

However, recall that the magnetic field of this kind of equilibrium points in the $-\hat{\theta}$ direction instead of the + direction as indicated in the above. Incorporating this additional negative sign we then arrive at,

$$u_r = g_0\frac{\omega_{cj}}{\nu^2 + \omega_{cj}^2} = v_{wf}^{(coll.)}(r) \tag{232}$$

From the above it is apparent that,

$$\nu >> \omega_{cj} \implies u_r \to 0 \tag{233}$$

$$\nu << \omega_{cj} \implies u_r \implies \frac{g_0}{\omega_{cj}} \tag{234}$$

Leaving aside the exact form of ν for the moment from this over-simplified picture we can imagine a potentially viable scenario for this application whereby highly-collisional regions in a plasma made from a broken-down pollutant gas stream give rise to appreciable quantities of pollutant ions that remain confined in the plasma as long as they continue to react with the background. However, as these pollutant ions travel down the length of the pinch in a collisionless manner, i.e., on a timescale that is shorter than the collisional one, then they will find themselves travelling outwards at significantly large speeds as shown quantitatively in the article, and this permits the possibility of escape from the plasma if they are travelling fast enough for a small enough pinch.

Resistive Drifts

Waterfall drifts are not the only radial drifts possible. When a finite resistivity is introduced then an Ohmic electrostatic field arises from the axial plasma current density,

$$E_z = \eta_\perp J_z \tag{235}$$

which leads to a plasma drift of the form,

$$\vec{v}_\eta = \frac{1}{q_s}\frac{q_s E_z \hat{z} \times B_\theta \hat{\theta}}{B^2} \tag{236}$$

Again, the azimuthal magnetic field has a lurking negative sign in the definition so we arrive at,

$$\vec{v}_\eta = \eta_\perp \frac{J_z}{B_\theta}\hat{r} \tag{237}$$

The above expression admits the possibility for the existence of a vortical configuration which has a large axial current near the axis of the pinch, while still possessing a sufficiently small magnetic field thereby exciting large radial drifts in this region. One quantitative example of this is shown in the main article.

Unsteady Bennett Vorticity

From the perspective of mathematical physics, it is worth exploring possible extensions to the equilibrium which permit its study in an unsteady context, as well as a non-exhaustive list of systems for which the study would potentially be fruitful. Two systems which exemplify the properties that would potentially lead to fruitful study are the Korteweg-de-Vries[**?**] (KdV),

$$\partial_t u + \partial_z^3 u - 6u\partial_z u = 0 \tag{238}$$

and Kuramoto-Sivashinsky[**?**] (KS) equations, either in 1D,

$$\partial_t u + \partial_z^2 u + \partial_z^4 u + \frac{1}{2}(\partial_z u)^2 = 0 \tag{239}$$

or the fully biharmonic 3D system,

$$\partial_t u + \nabla^2 u + \nabla^4 u + \frac{1}{2}|\nabla u|^2 = 0 \tag{240}$$

What makes these systems exemplary is their first-order, linear temporal structure while possessing spatial structures which admit highly nonlinear effects. The relaxability of the axial symmetry is a boon in this regard as modulations to one or more of the parameters necessary to describe the vortex can be used to connect this unsteady character to, e.g., the axial transport of the vortex, or even the full 3D flow field in the case of KS. This observed relaxibility is also the basis for the form of the spatial derivatives presented in the above 1D equations. In the KS case we can derive a consistent axisymmetric, cylindrical system from the fully 3D, biharmonic system. In the KdV case we can also treat the radial direction in 1D, but this must be done delicately due to the cylindrical geometry implied by the radius, and further work in this regard is outside the scope of this supplement.

The main question in regard to an unsteady vortex is how the time-dependence is modulated in the underlying flow field. There are of course other nonlinear systems beyond the aforementioned which exist, and some which possess higher-order time dependencies while still retaining the linear nature which is a boon to any analytical campaign attempting to study the consequences of the spatial nonlinearity over one involving nonlinear evolution terms. A treatment of any system in the manner described here is beyond the scope of this supplement, and this commentary is only meant to discuss possible means by which to explore the unsteady nature of Bennett vortices.

A postscript to note in this regard is the application of this functional form to the modelling of a time-only signal, e.g.,

$$I(t) = I_0 \frac{t^2}{(t + C_{B,I})^2} \tag{241}$$

to provide a ramp-law. The ramp-time constant $C_{B,I} \neq C_{B,T}$ for reasons of dimensional mismatch so this will need to be determined from the boundary conditions of the system. Continuing in this postscriptoral vein, the time-signal can be also attached separately to the fundamental Bennett form,

$$u_z(r,t) = A(t) u_{z,0} \frac{r^2}{(r + C_{B,T})^2} \tag{242}$$

to describe a vortex-modulated amplitude excitation. This is perhaps one of the most tractable ways to study the unsteady regime as the separability lends the temporal character cleanly to any nonlinear system. This gives different ramp-forms that are naturally attached to the aforementioned nonlinear PDEs,

$$KdV: \; \dot{A} u_B(z) + (\partial_{zzz} u_B) A - 6 u_B u_B' A^2 = 0 \tag{243}$$

$$KS: \; \dot{A} u_B(z) + A(u_B'' + u_B'''') + A^2 \frac{1}{2} u_B'^2 = 0 \tag{244}$$

$$KS(3D): \dot{A} u_B(r) + A\left(\frac{1}{r}\frac{\mathrm{d}(r u_B')}{\mathrm{d}r} + \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r} r (\nabla^2 u_B)'\right) + A^2 \frac{1}{2} u_B' u_B'^* = 0 \tag{245}$$

Bennett-Grad-Shafranov Vorticity

In toroidal coordinates the analogous axisymmetric ideal MHD system is given by the Grad-Shafranov equation, which has the same form as a Hick's equation for an axisymmetric inviscid fluid,

$$\frac{\partial^2 \psi}{\partial r^2} - \frac{1}{r}\frac{\partial \psi}{\partial r} + \frac{\partial^2 \psi}{\partial z^2} = -\mu_0 r^2 \frac{\mathrm{d}p}{\mathrm{d}\psi} - \frac{1}{2}\frac{\mathrm{d}F^2}{\mathrm{d}\psi} \tag{246}$$

with the stream function, ψ, being introduced so that the incompressibility of the axisymmetric system,

$$\nabla \cdot \vec{u} = 0 \tag{247}$$

is built in to the formulation via,

$$\vec{u} = \frac{1}{r} \nabla \psi \times \hat{\theta} \tag{248}$$

which gives,

$$u_r = -\frac{1}{r}\frac{\partial \psi}{\partial z} \tag{249}$$

$$u_z = \frac{1}{r}\frac{\partial \psi}{\partial r} \tag{250}$$

In the case of the cubic, pure-flow vortices that we have been studying the stream function is supplied fundamentally by a toroidal flow that is locally axial,

$$\psi(r,z) = \psi(0,z) + u_{z,0}\int_0^r \frac{r'^3}{(r'+C_{B,T})^2}dr' \tag{251}$$

The accompanying radial flow is then given entirely by the free function along the pinch axis,

$$u_r = -\frac{1}{r}\left(\frac{\partial\psi(0,z)}{\partial z}\right) \tag{252}$$

which introduces problems with regularity unless the stream function goes to zero sufficiently fast enough near the axis, or is treated as being separable into a form which annihilates the singularity, e.g.,

$$\psi(r,z) \sim r^2 F(z) + \psi_0(z) \tag{253}$$

however, the above canonical form is still inconsistent with a Bennett vortex for the locally axial flow. An additional gauge must be added so that,

$$u_z(r) = \frac{1}{r}\frac{\partial\psi}{\partial r} \tag{254}$$

$$= \frac{1}{r}\left(2rF(z) + \frac{\partial\psi_B(r,z)}{\partial r}\right) \tag{255}$$

$$= u_B(r) \tag{256}$$

which implies that,

$$\psi_B(r,z) = \psi_B(0,z) - r^2F(z) + \int_0^r r' u_B(r')dr' \tag{257}$$

so that the radial flow field is,

$$u_r = -\frac{1}{r}\left(r^2F'(z) + \psi_0'(z) + \psi_B(r,z)_{,z}\right) \tag{258}$$

$$= -\frac{1}{r}\left(\psi_0'(z) + C'(z)\right) \tag{259}$$

$$= -\frac{1}{r}\tilde{\psi}_0'(z) \tag{260}$$

where,

$$\tilde{\psi}_0(z) = \psi_0(z) + \psi_B(0,z) \tag{261}$$

$$= \psi_0(z) + C(z) \tag{262}$$

Interestingly, the structure of the breathing mode does not influence the radial flow field when the toroidal flow is locally describable by that of the given Bennett vortex. Rather, the radial flow field depends entirely on two free functions, and its regularity on their cancellation when an axial derivative is taken.

Magnetic Reconnection

When a finite resistivity is incorporated into the plasma dynamics, more than just radial drifts develop. Specifically, the magnetic field can also assume an unsteady, and diffusive, character which arises when field line topologies are no longer "frozen-in" to the plasma, e.g., near regions of large magnetic shear. Bennett vortices naturally have such a region occur in the equilibrium as the magnetic shear remains small until the flow shear layer develops the plasma close to its edge state, and then the onset of relatively large amounts of enclosed current across a thin region results in the onset of relatively large amounts of magnetic field amplitude across a thin region, i.e., a current-sheet like structure.

In the case of a finite resistivity, governance of the magnetic field expands in scope as previously described, obtaining,

$$\frac{\partial \vec{B}}{\partial t} = \nabla \times (\vec{u} \times \vec{B}) + \eta \nabla^2 \vec{B} \tag{263}$$

The structure of the magnetic field's unsteady, and diffusive character depends on the underlying form of the vortex. We have studied three steady flow forms so far,

$$\vec{u} = u_z(r)\hat{z} \tag{264}$$

$$\vec{u} = u_r(r, z)\hat{r} + u_z(r)\hat{z} \tag{265}$$

$$\vec{u} = u_\rho(\rho, \phi)\hat{\rho} + u_\phi(\rho, \phi)\hat{\phi} = u_z(r)\hat{z} \tag{266}$$

as well as commented on the possibility of attaching various temporal modulations to the flow field pattern by giving an unsteady character to some underlying plasma parameter that is encapsulated by the shear layer positioning,

$$\vec{u}(r, t) = u_z(r, t)\hat{z} = u_{z,0}\frac{r^2}{(r + C_{B,T}(t))^2}\hat{z} \tag{267}$$

or by an amplitude signal,

$$u_z(r, t) = A(t)u_B(r) \tag{268}$$

and whose consequences can be readily studied in the context of various nonlinear systems, e.g., KdV, or KS, as described previously. While these extensions provide natural routes to studying the evolution of a diffusive magnetic field, a quantitative treatment of reconnection requires resistive, and kinetic effects which are outside the scope of this investigation. This brief note is merely to highlight the numerous possibilities that the natural geometric structure of the theory furnishes for settings in which these processes may arise.

Ball Lightning

It is also ventured that perhaps ball lightning could be a Bennett Vortex. In fact, this is not just wild speculation as observations of the phenomenon are frequently characterized by reports of it moving slowly in a single direction[**?**]. This matches both with the purely axial flow of a Bennett Vortex, and also the requirement that a macroscopic, "squat" pinch

$$r_p > L \tag{269}$$

has for being small-$C_{B,T}$. Namely, that the flow speed must be small!

$$C_{B,T} = C_{B,u} n_0 u_{z,0}^2 \frac{r_p^3}{T_p} \tag{270}$$

For a terrestrial vortex, let us think of the ball lightning as being a magnetized glob of charge which possesses mass, momentum, energy, and thermal power as residues of a lightning strike. The environment will absorb some of this energy, and the rest will be put into sustaining a plasma for as long as the conditions afford.

The sustainment of ball lightning for such long periods can be explained by the formation of a plasma with Bennett Vortices in it from the stray thermal and electrical power left over in the aftermath of a lightning strike. This is perhaps a creative tangent to attempt and make, but the security of a shear-flow stabilized Z-Pinch with the thermal power still on affords a means by which the diffusive magnetic equilibrium could sustain itself for macroscopic timescales if there was sufficient thermal energy to begin with to justify its existence locally. Ball lightning would also have some stray electrical power remnant leftover from the lightning strike.

The confinement time, and properties of this plasma can be calculated. When making calculations, it is best to stick to a small-$C_{B,T}$ plasma as this will not require you to first solve for the $u_{z,0}$ roots. If we consider a plasma where,

$$n_0 = 10^{25}\ [m^{-3}] \tag{271}$$

$$u_{z,0} = 10^{-3}\ [m\ s^{-1}] \tag{272}$$

$$r_p = 1\ [m] \tag{273}$$

$$T_p = 1000\ [degK] \tag{274}$$

then the resulting confinement time is,

$$\tau_E = 75.8\ [s] \tag{275}$$

a value which is in good accord with observations where confinement times range up to a minute. The parameter $C_{B,T} = 1.46 * 10^{-6}$ is small, and the total power of $P = 164\ [MW]$ is a very sensible number for the residual power of a lightning strike which could involve gigawatts of electrical power being discharged through the air.

To connect this more strongly to the formation process for ball lightning in the aftermath of a lightning strike, then viscous, and thermal plasma chemistry effects would also need to be incorporated as the air becomes superheated and begins to drag on the plasma, with the energy leading to excitation of the plasma species, and possible chemical reactions. The lightning could potentially form in an evacuated pocket left behind when the current is done flowing into the ground. If not all of this current was able to be absorbed by the locality of the strike, then some of it would be ejected into this pocket with the leftover energy of the strike, alongside material from the ground which will advect with the plasma if it becomes ionic, thereby naturally being confined by the strong edge magnetic field.

Bennett-Tipler Vortices

The circumstances under which the investigation of the Shumlak criterion proceeded in order to yield the structure of these vortices are somewhat analogous to that of the emergence of Closed Timelike Curves (CTCs) in a Tipler cylinder where sufficiently strong rotation in an infinitely long cylinder causes light cones to tilt back over on themselves and close. While the present work is entirely within the context of non-relativistic, classical, ideal magnetohydrodynamics the analogy between strong vorticity and frame-dragging effects suggests a broader perspective by which strong rotational flow fields organize structure.

In light of the Casimir Effect, which shows that a negative energy density appears between two conducting plates that are sufficiently close together due to the reduction of certain electromagnetic modes in the interstitial quantum vacuum, it bears investigating the relativistic consequences that two infinitely long rotating concentric cylinders of plasma have for this interstitial vacuum. It is important to emphasize that these negative energy densities arise from quantum boundary conditions on the fluctuation fields, and not classical plasma dynamics. Regardless, the study of highly sheared, rotating plasma configurations in this state provides a conceptual bridge towards understanding how energy distributions are influenced by rotating structure in more general field-theoretic settings.

The author's primary motivation for making this suggestion, beyond to expand the frontiers of gravitational and plasma physics, is to consider whether such configurations could, in principle, inform discussions of exotic energy conditions in relativistic spacetimes. Any potential relevance to constructs such as traversable wormholes would require a fully quantum field-theoretic treatment unified, and consistently coupled with the plasma dynamics presented in this article, and lies outside the scope of the present supplementary material.

Relativistic Bennett Vorticity

A fully relativistic theory of Bennett vorticity involves two levels of treatment. In the simplest flat spacetime treatment we cannot study the effect of mass on the local curvature of a configuration, and so to study the dynamics of vortices within a mass-dominated context we would need to extend our treatment to the second level, namely, that of the domain of general relativity. This would need to happen before we could study the boundary between filamentary plasmas and the gravitating, vortical systems which are observed together at the nodes of this filamentary network in the context of the cosmic web. This is beyond the scope of the current work, and neither shall we start from a first principles framework for the relativistic conservation of mass, momentum and energy, based on energy-momentum, electromagnetic stress, and material flux throughout a metered spacetime.

Instead, we can turn our attention to the electron inertia, which in the idealized MHD system we have studied heretofore is considered trivial. As the electron speed increases to relativistic scales we must treat the momentum as,

$$p_e = \gamma_z m_e u_z \tag{276}$$

where,

$$\gamma_z(r) = \frac{1}{\sqrt{1 - \frac{u_z^2(r)}{c^2}}} \tag{277}$$

If we take n_0 to be the proper density, i.e., the density measured in the electron frame then the relativistic contraction of space requires the plasma current density to be corrected,

$$J_z = q\gamma_z(r) n_0 u_z(r) \tag{278}$$

and the kinetic energy density,

$$\varepsilon_{K,e} = (\gamma_z - 1)\gamma_z n_0 m_e c^2 \tag{279}$$

In regimes where $\gamma >> 1$ the relativistic inertia of the electrons then require modifications to the pressure balance, magnetic field, and plasma current density.

Kadomtsev Stability

A Z-pinch can be made stable to the $m = 0$ mode by tailoring the pressure profile in a certain way as B.B. Kadomtsev discovered[?]

$$-\frac{\mathrm{d}\ln(p)}{\mathrm{d}\ln(r)} \geq \frac{4\Gamma}{2+\Gamma\beta} \tag{280}$$

which can be manipulated into the form,

$$B^2 \leq -\frac{\mu_0 \Gamma p}{2\Gamma + \frac{\mathrm{d}\ln(p)}{\mathrm{d}\ln(r)}} \frac{\mathrm{d}\ln(p)}{\mathrm{d}\ln(r)} \tag{281}$$

for $\beta = \frac{2\mu_0 p}{B^2}$. For $T = C_T r^3$ the above can be written as,

$$B^2 + \frac{3\mu_0 \Gamma (n_0 k_B C_T)^2}{2\Gamma + 3n_0 k_B C_T} r^3 \leq 0 \tag{282}$$

evidently leaving only one possible path forward in pursuing the above line of attack, since both B^2, and the cubic term are positive-definite for $C_T > 0$. Namely, finding roots to the equation,

$$B^2(r) + \frac{3\mu_0 \Gamma (n_0 k_B C_T)^2}{2\Gamma + 3n_0 k_B C_T} r^3 = 0 \tag{283}$$

which for the specific case of $T = C_T r^3$ can be represented by,

$$\frac{\mu_0^2 e^2 n_0^2 u_{z,0}^2}{2r^2(r + C_{B,T})^2} f^2 + \frac{3\mu_0 \Gamma (n_0 k_B C_T)^2}{(2\Gamma + 3n_0 k_B C_T)} r^3 = 0 \tag{284}$$

The highest-order in f^2 is sixth, however, the denominator attached to f^2 has fourth-order terms lurking which must be multiplied through, leading to,

$$\begin{gathered} r^7 + \mu_0 e^2 u_{z,0}^2 \left(\frac{\Gamma + 3n_0 C_T k_B}{6\Gamma k_B^2 C_T^2}\right) f^2(r, C_{B,T}) \\ + 2C_{B,T} r^6 + C_{B,T}^2 r^5 = 0 \end{gathered}$$

the roots of this equation would give relationships amongst the parametric values of the model that represent states where the Kadomtsev condition is exactly met. No analytic formula is, or can be, known for root-finding with arbitrary coefficients for polynomials of degree greater than or equal to 5[?]. Therefore, roots to Equation (284) would need to be found numerically. This computation is of interest as these roots describe Bennett Vortices which are also at the very limit of what Kadomtsev stability to the $m = 0$ mode would accept.

Sawteeth

Sawteeth naturally emerge in chains of these vortices. A single tooth looks like,

$$\triangle(r) = \begin{cases} \frac{u_0}{r_p} r & \text{if } 0 < r \leq r_p \\ -\frac{u_0}{r_p} r + 2u_0 & \text{if } r_p < r < 2r_p \\ 0 & \text{else} \end{cases} \tag{285}$$

Writing this as a chain of core pure-flow cubic vortex, and bulk, negative, pure-flow cubic vortex we have,

$$u(r) = \begin{cases} u_{z,0}\frac{r^2}{(r+C_{B,T})^2} & \text{if } 0 < r \leq r_p \\ u_0 - u_{z,0}\frac{r^2}{(r+C_{B,T})^2} & \text{if } r_p < r \leq 2r_p \\ 0 & \text{else} \end{cases} \tag{286}$$

which is equivalent to,

$$u(r) = u_z^{(2)}(r)\square(r) + u_z^{(2,-)}(r - r_p)\square(r - r_p) \tag{287}$$

where,

$$\square(r) = \begin{cases} 1 & \text{if } 0 \leq r \leq r_p \\ 0 & \text{else} \end{cases} \tag{288}$$

At the pinch radius we have,

$$u_z^{(2,-)}(r_p - r_p) = u_z^{(2,-)}(0) = u_0 \tag{289}$$

$$u_z^{(2)}(r_p) = u_{z,0}\frac{r_p^2}{(r_p + C_{B,T})^2} \tag{290}$$

Equating these two we have,

$$u_0 = u_{z,0}\frac{r_p^2}{(r_p + C_{B,T})^2} \tag{291}$$

where the ratio of the core speed to the flow modulus of the second vortex naturally emerges as a key parameter,

$$Ur = \frac{u_0}{u_{z,0}} \tag{292}$$

However, we can study the structure of the normalized system instead of having to pin down a value for the above,

$$\tilde{u}_z^{(2,-)}(0) = \frac{u_z^{(2,-)(0)}}{u_0} = 1 \tag{293}$$

and,

$$\tilde{u}_z^2(\phi_p) = \frac{u_z(r_p)}{u_{z,0}} = \frac{\phi_p^2}{(\phi_p + 1)^2} \tag{294}$$

with,

$$\phi_p = \frac{r_p}{C_{B,T}} \tag{295}$$

Independent of the flow ratio number we then have the requirement that,

$$\phi_p^2 = (\phi_p + 1)^2 \tag{296}$$

$$\implies 2\phi_p = -1 \tag{297}$$

$$\therefore \phi_p = -\frac{1}{2} \tag{298}$$

which is impossible to obtain in the positive half-chord. This implies that an arbitrary 2-chain of vortices will inevitably construct a non-uniform sawtooth as the lack of a physically-realizable match at the boundary of the vortices means that either one, or both, of the vortices will undershoot their target depending on the exact structure of the system. Naturally, this introduces a jump discontinuity into the system at the interface that suggests a mechanism by which shocks can naturally form.

L-H Mode Transition

The existence of a high confinement "H" mode in tokamaks is coincident with the development of a transport barrier at the edge of the plasma that appears when the plasma is sufficiently heated. This transport barrier is further accompanied by the observation of enhanced thermodynamic gradients in the plasma. The current physical explanation for this process is that sheared $E \times B$, and zonal flows, alongside large edge plasma pressure gradients leads to suppression of the turbulent transport, but the exact self-organization of this system is unexplained.

This remains an active area of investigation, and the structure of the Bennett vortex family is highly suggestive in this context. As such, it could serve as a useful model which provides a local analytic form for describing the shear-layer structure that appears in this transition. In particular, the coexistence of these structures with large thermal edge gradients suggests a possible route by which sufficiently strong heating can drive the plasma into a shear-dominated, transport-suppressed regime.

A quantitative investigation of the relevant growth rates, balanced against the turbulent decorrelation of the sheared plasma dynamics, is beyond the scope of this present work, and left for future study.

Thermal Lifetime & Vortex Plume Structure

The implication that each vortex has a finite thermal lifetime,

$$\tau_E = \frac{3}{2} \frac{p_0 V_p}{\int_{A_L} \vec{q} \cdot d\vec{A}} \tag{299}$$

and its properties as a shear-flow stabilized Z-pinch means that the process of an $m = 0$ mode quenching the pinch at the end of its lifetime can interpreted kinematically in reverse as watching plume expansion from a compact core. This observation is useful in the context of computational applications, e.g., fusion propulsion, waterfall reactors, cosmology, etc. for obtaining an insight into the state of the magnetohydrodynamic fluid, especially at the surface of the compressive necking process, as an adjoint construction to it. In the context of engineering studies, this would give numerical insight into the aspect ratio of the simulated process, defined by the pinch radius near the collapse point over the equilibrium pinch radius,

$$R^* = \frac{r_{neck}}{r_p} \tag{300}$$

the specific way to obtain r_{core} computationally is outside the scope of this article, and left for a future work.

The structure of the thermal lifetime is of the utmost importance for any application of this profile, and so is the usage of this to extend the thermal lifetime of reactors, as well as to better understand the physics behind the process of an ideal, shear-flow stabilized thermal lifetime captured by a perpendicular thermal conductivity. The self-similarity of the equilibrium family solutions, meaning the class which retain the same locating of the shear layer, and same flow velocity, relative to each other, are identical in flow pattern, and can therefore have their plasma conditions varied, in particular plasma number density and temperature, and obtain the exact same flow pattern for a higher temperature and a larger density, as well as the converse.

The presence of "inverse" parabolic temperature profiles in Zap DD shear-flow stabilized Z-pinch fusion plasma[**?**] experiments, i.e., the "weak" form of the vortex, also motivates investigating the consequences of the parabolic temperature for the impact on thermal lifetime. Especially, to find ways to extend it naturally or to provide a setting for the state of the plasma during the collapse / expansion process to study the conditions of the plasma during an attempt to resuscitate the pinch, or just naturally as one part of the pulsed power cycle for a whole-device model where requirements for pinch current and repetition rate define the fusion gain, and adiabaticity of the fusion plasma process.

The search for a maxima of pinch lifetime occurs in a 4D space of $(n_0, u_{z,0}, r_p, T_p)$, and complex solutions to the root of a flow constant produce the same $C_{B,T}$ plasma relative to a real flow with the same magnitude, so their existence does not interfere with the definition of a thermal lifetime. One approach to resolving this space is to first fix the flow constant, which is a natural aid to an experimental investigation, and required for exactly self-similar solutions. However, the calculation for the pressure depends on logarithmic terms that introduce numerical instabilities into the solution, and there is not an immediately apparent fix as when this problem arose in making drift speed calculations.

There are also numerous ways to treat the thermal lifetime that do not hinge on calculating the representative pressure in the above manner. There are also representative pressures for bulk vortices, and other orders of vortex that have been presented in this supplement, which establish how the norm of this could be taken to explore additional avenues of thermal lifetime study. Experimental values may also of course be taken as representative for the plasma pressure.

Extended Vorticity

The Shumlak criterion is what judges a plasma with axial flow to be shear-flow stabilized or not. In the case of the Z-pinch equilibrium the vorticity is,

$$\vec{\omega} = \nabla \times \vec{u} = -\frac{\mathrm{d}u_z}{\mathrm{d}r}\hat{\theta} \tag{301}$$

so that interestingly the Shumlak criterion can be written in this case as,

$$|\vec{\omega} \cdot \hat{\theta}| = |\omega_\theta| > 0.1kV_A \tag{302}$$

or more generally,

$$|\vec{\omega}| > 0.1kV_A \tag{303}$$

because the relevant shear naturally arises on the LHS, so each of these interpretations are equivalent in this specific case. This natural occurrence is highly suggestive of deeper mathematical structure, but care should be taken when interpreting the implications because in general the above is not true. It is only true when considering a pure Z-pinch.

The most rigorous implication is that a wide variety of shear-flow stabilized Bennett-Shumlak vortices is possible, as any vorticity that possessed sufficient radial shear in the axial velocity can have cross-vorticities that exist alongside this shear-flow stabilized character which do not compromise it for the original forms presented in this work. For example, an axisymmetric system where $\frac{\partial}{\partial\theta} \to 0$,

$$\vec{\omega} = -\frac{\partial u_\theta}{\partial z}\hat{r} + (\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r})\hat{\theta} + \frac{1}{r}\frac{\partial(ru_\theta)}{\partial r}\hat{z} \tag{304}$$

that can also be taken as swirlless,

$$\vec{\omega} = (\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r})\hat{\theta} \tag{305}$$

or relaxed entirely,

$$\vec{\omega} = (\frac{1}{r}\frac{\partial u_z}{\partial \theta} - \frac{\partial u_\theta}{\partial z})\hat{r} + (\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r})\hat{\theta} + (\frac{1}{r}\frac{\partial (r u_\theta)}{\partial r} - \frac{1}{r}\frac{\partial u_r}{\partial \theta})\hat{z} \tag{306}$$

The growth of swirl from the fundamental physics of the cylindrical equilibrium has not been addressed yet so we can either focus on the swirlless, axisymmetric form in a cylindrical basis, or study the implications for the expression of this vector in a spherical one.

Let us study these spherical implications first, where the axial plasma flow becomes,

$$\vec{u} = u_\rho(\rho,\phi)\hat{\rho} + u_\phi(\rho,\phi)\hat{\phi} \tag{307}$$

giving a vorticity,

$$\vec{\omega} = (\frac{1}{\rho\sin(\theta)}\frac{\partial u_\rho}{\partial \phi} - \frac{1}{\rho}\frac{\partial(\rho u_\phi)}{\partial \rho})\hat{\theta} \tag{308}$$

demonstrating that the introduction of swirl does not arise from any consequence of the spherical description of a pure Z-pinch. However, it does suggest that the azimuthal vorticity can naturally be thought of as a reflection of two terms in this equilibrium. This is a meaningful observation to make as it suggests that in a quasi-equilibrium pinch axial gradients in the radial in/outflow will lead to an increase in azimuthal vorticity. In an equilibrium pinch of the kind we have studied, these radial flows will formally have no such gradients. The addition of an unsteady character to the system admits the possibility for the growth of strong electric fields in cross directions which could give rise to substantial swirling, or in/out-flowing components.

Another observation to make is that the relaxation of axial symmetry in the cylindrical basis, e.g.,

$$\vec{u}(r,z) = u_r(r,z)\hat{r} + u_z(r)\hat{z} \tag{309}$$

does not introduce any fundamentally new structure to the spherical description,

$$\begin{aligned}\vec{u}(r,z) &= (u_r(\rho,\phi)\sin\phi + u_z(\rho,\phi)\cos\phi)\hat{\rho} + (u_r(\rho,\phi)\cos\phi - u_z(\rho,\phi)\sin\phi)\hat{\phi} && (310)\\ &= \vec{u}(\rho,\phi) && (311)\end{aligned}$$

although it does suggest further avenues to explore the possibility of flow stagnation on these directions. This is an important path to explore because an instant of stagnated flow in the polar direction will lead to the appearance of a purely radial flow,

$$u_\rho = 2u_r\sin\phi \tag{312}$$

defined by,

$$u_z = u_r\tan\phi \tag{313}$$

We can also study the conditions for (spherical) radial stagnation, which would leave us with a flow that appears to be completely polar,

$$\begin{aligned}u_\rho &= 0 && (314)\\ \therefore u_r\sin\phi &= -u_z\cos\phi && (315)\\ \implies u_z &= -u_r\tan\phi && (316)\\ \therefore u_\phi &= u_r\cos\phi + u_r\sin\phi\tan\phi && (317)\\ &= u_r\frac{\cos^2\phi + \sin^2\phi}{\cos\phi} && (318)\\ &= \frac{u_r}{\cos\phi} && (319)\end{aligned}$$

which is interesting because it suggests that in this regime certain polar angles will naturally lead to a singular explosion in the velocity which are not present in the converse situation.