Another Shear-Flow Stabilized Z-pinch Equilibrium

Matt Russell

August 2, 2026

Introduction

The idea of shear-flow stability starts with the Shumlak-Hartman criterion,

$$\frac{\mathrm{d}u_z}{\mathrm{d}r} > 0.1kV_A \tag{1}$$

where the shear on the LHS of Equation (1) refers to the axial MHD flow shear, and the quantities on the RHS are the Alfven speed,

$$V_A = \frac{B}{\sqrt{\rho\mu_0}} \tag{2}$$

and the fundamental axial wavenumber,

$$k = \frac{2\pi}{L} \tag{3}$$

For a given axial MHD flow profile, the task of determining whether it counts as a shear-flow stabilized Z-pinch depends on satisfying Equation (1). The plasma current density can be directly linked to the MHD flow in certain situations. For a hydrogen-electron plasma this linkage exists when one species can be treated as stationary compared to the other. If the ions are treated as stationary, then the electrons need to be suitably relativistic to avoid a pre-factor from the reduced mass. The case of ideal reduction from a multifluid plasma will not be considered here.

Sonnet-5 Vortex

Claude Sonnet-5 was asked to construct a shear-flow stabilized Z-pinch equilibrium. Previously, the same task has been successfully achieved by GPT-5.5. When asked, that frontier model chose to take the cubic, pureflow Bennett vortex,

$$u_z(r) = u_{z,0}\frac{r^2}{(r+C)^2} \tag{4}$$

where,

$$C \simeq 10^{-22} n_0 u_{z,0}^2 r_p^3 T_p^{-1} \tag{5}$$

and remove the the quadratic cross-term from the denominator. Sonnet-5 used a different strategy. This frontier model instead took the Bennett plasma number density,

$$n(r) = \frac{n_0}{(1+\frac{r^2}{C^2})^2} \tag{6}$$

and attached a linear flow with constant shear,

$$u_z^{S5}(r) = u_{z,0}\frac{r}{C} \tag{7}$$

which yields a plasma current density of the form,

$$J_z(r) = en(r)u_z(r) = en_0 u_{z,0}\frac{r}{C}\frac{1}{(1+\frac{r^2}{C^2})^2} \tag{8}$$

It will not be evaluated here whether a self-similar solution can be obtained between this current density and a Bennett vortex current density as that is outside the scope of this work which is to present another shear-flow stabilized Z-pinch equilibrium.

To that end we investigate whether a tractable magnetic field is obtained from Ampere's Law,

$$B_\theta = \frac{\mu_0 J_0 C^3}{r}\int_0^r \frac{r'^2}{(r'^2+C^2)^2}dr' \tag{9}$$

The integral is tractable, and can be evaluated via trig-substitution, or derivative identity using,

$$\frac{\mathrm{d}}{\mathrm{d}r'}\left(\frac{r'}{r'^2+C^2}\right) = \frac{C^2-r'^2}{(r'^2+C^2)^2} \tag{10}$$

Trig substitution is more straightforward. Substituting,

$$r' = C\tan(\theta) \tag{11}$$

we have,

$$\begin{aligned}
I(r) &= \int_0^r \frac{r'^2}{(r'^2+C^2)^2}dr' & (12)\\
&= \int_0^{\tan^{-1}(\frac{r}{C})} \frac{C^2\tan^2(\theta)}{(C^2\tan^2(\theta)+C^2)^2}C^2\sec^2(\theta)d\theta & (13)\\
&= \frac{1}{C}\int_0^{\tan^{-1}(r/C)} \frac{\tan^2(\theta)}{\sec^2(\theta)}d\theta & (14)\\
&= \frac{1}{C}\int_0^{\tan^{-1}(r/C)} \sin^2(\theta)d\theta & (15)\\
&= \frac{1}{2C}\left[\tan^{-1}(r/C) - \frac{Cr}{r^2+C^2}\right] & (16)
\end{aligned}$$

Yielding an azimuthal magnetic field of the form,

$$B_\theta(r) = \frac{\mu_0 J_0 C^2}{2}\left[\frac{1}{r}\tan^{-1}(r/C) - \frac{C}{r^2+C^2}\right] \tag{17}$$

It is interesting to compare this to the magnetic fields obtained from a cubic, pureflow Bennett vortex, and a shear flow profile derived from the cubic, pureflow Bennett vortex by truncating away the quadratic cross-term. These fields are quoted below in respective order,

$$B_\theta^{(BV)}(r) = \mu_0 J_0 \frac{f(r,C)}{2r(r+C)} \tag{18}$$

$$f(r,C) = r^3 - 3r^2C - 6rC^2(1+\ln\left(\frac{C}{r+C}\right)) - 6C^3\ln\left(\frac{C}{r+C}\right) \tag{19}$$

$$B_\theta^{(G5)}(r) = \frac{J_0\mu_0}{2r}\left(r^2 + C_{B,T}^2\ln\left(\frac{C_{B,T}^2}{r^2+C_{B,T}^2}\right)\right) \tag{20}$$

There is an interesting difference in the transcendental structure to the magnetic field. In the case of previous shear-flow stabilized Z-pinches this transcendental structure is logarithmic, whereas here it is an arctangent instead. Both of the magnetic fields written above can be shown to satisfy Shumlak-Hartman for an arbitrarily small plasma radius via L'Hopitals rule as both of these magnetic fields go to zero there. Far from the pinch axis both of these magnetic fields asymptote to linear behavior.

Another difference is in the number density. The Bennett profile remains attached to the plasma number density here so we have to reckon with it when evaluating the Shumlak-Hartman criterion. The expression of interest to evaluate is then,

$$\frac{u_z'\sqrt{n(r)}}{B_\theta(r)} > \frac{\pi}{5L\sqrt{m\mu_0}} \tag{21}$$

The LHS of the above reduces to,

$$\frac{u_z'\sqrt{n(r)}}{B_\theta(r)} = \frac{2u_{z,0}\sqrt{n_0}}{\mu_0 J_0 C}\frac{r}{(r^2+C^2)\tan^{-1}(r/C) - rC} \tag{22}$$

Moving everything to the RHS that is a constant which can be moved we have,

$$\frac{r}{(r^2+C^2)\tan^{-1}(r/C) - rC} > \frac{\pi}{5L\sqrt{m\mu_0}}\frac{\mu_0 J_0 C}{2u_{z,0}\sqrt{n_0}} \tag{23}$$

$$> \frac{\pi}{10L}eC\sqrt{\frac{\mu_0 n_0}{m}} \tag{24}$$

Let us study the limits of the LHS. In the ideal MHD case the RHS can be asymptotically reduced to a value of zero from the limit $L \to \infty$. However, for any non-trivial plasma state we find that close to the axis the LHS quantity diverges. After application of L'Hopital's rule we have equivalently,

$$\lim_{r \to 0} \frac{1}{2r \tan^{-1}(r/C)} \to \infty \tag{25}$$

For the large plasma radius limit we instead have,

$$\lim_{r \to \infty} \frac{1}{2r \tan^{-1}(r/C)} \to 0 \tag{26}$$

so that this pinch satisfies the weak-form of the Shumlak-Hartman criterion. Therefore, we can say that this equilibrium can be thought of as shear-flow stabilized everywhere in the pinch radius. Close to the pinch axis this is satisfied automatically for any non-trivial problem. Far away from the pinch axis, this is satisfied weakly, meaning when we consider the consequences of taking $L \to \infty$.

The last order of business is to compute the plasma pressure, however that is a daunting task based on integrating,

$$\frac{\mathrm{d}p}{\mathrm{d}r} = -J_z B_\theta \tag{27}$$

so we will stop this brief note here for the moment as we have proven that this flow is a shear-flow stabilized Z-pinch.