The Core Pressure of a Cubic Bennett Vortex

Matt Russell

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Consider the cubic, pureflow Bennett, or Bennett-Shumlak-Hartman, vortex. This is a shear-flow stabilized Z-pinch equilibrium given by,

$$u_z(r) = u_{z,0}\frac{r^2}{(r+C_{B,T})^2} \tag{1}$$

The magnetic field obtained when treating this as the flow supporting the plasma current density is,

$$B_\theta(r) = \frac{en_0\mu_0 u_{z,0}}{2r(r+C_{B,T})}f(r,C_{B,T}) \tag{2}$$

as a change of sign in the current density will only result in a change of direction in the magnetic field. It is valid to treat the system in this manner under certain conditions. We are interested in the plasma pressure. The ideal gas law suggests it will be of the form,

$$p_{IGL}(r) = n_0 k_B T(r) \tag{3}$$

however this is inconsistent compared to the form obtained from the momentum balance,

$$p(r) = p_0 - \int_0^r J_z B_\theta dr \tag{4}$$

The core pressure, p_0, is obtained from the boundary conditions. The plasma pressure at the edge is taken to be continuous with the vacuum so that,

$$p(r_p) = 0 \tag{5}$$

yielding,

$$p_0 = \int_0^{r_p} J_z B_\theta dr \tag{6}$$

Inserting,

$$J_z(r) = en_0 u_z(r) \tag{7}$$

into the above is valid for non-relativistic ions with electrons that are treated as stationary. This treatment is justified in an average sense for the ordering $\omega_{pe} >> \bar{\nu}_{ei}$ as the electrons will oscillate so fast from the ion frame that they appear to be located at their equilibrium point. This obtains,

$$p_0 = \frac{(en_0 u_{z,0})^2\mu_0}{2}\int_0^{r_p}\frac{r}{(r+C_{B,T})^3}f(r,C_{B,T})dr \tag{8}$$

The function from the magnetic field form is,

$$f(r, C_{B,T}) = r^3 - 3r^2 C_{B,T} - 6r C_{B,T}^2 (1 + \ln\left(\frac{C_{B,T}}{r + C_{B,T}}\right)) - 6C_{B,T}^3 \ln\left(\frac{C_{B,T}}{r + C_{B,T}}\right)$$

We can evaluate the pressure integral through u-substitution or by means of a CAS. Here, we choose CAS for brevity. This yields,

$$\int_0^{r_p} \frac{r}{(r+C)^3} f(r, C) dr = \frac{r_p(2C + r_p)}{2(C + r_p)^2}(-15C^2 - 12Cr_p + r_p^2) \tag{9}$$

$$+ \frac{6C^2}{2(C + r_p)} \ln\left(\frac{C}{C + r_p}\right)\left(-5C - 3r_p + (C + r_p)\ln\left(\frac{C}{C + r_p}\right)\right) \tag{10}$$

What happens if we have a bulk-flow vortex instead of a pureflow one? This question is relevant because bulk vortices appear in real plasmas. Then, we have the plasma current density,

$$J_z^{(\pm)} = en_0\left(u_0 \pm u_{z,0} \frac{r^2}{(r + C)^2}\right) \tag{11}$$

and the magnetic field,

$$B_\theta^{(\pm)}(r) = \frac{\mu_0}{r} \int_0^r r' J_z^{(\pm)} dr' \tag{12}$$

$$= \frac{en_0\mu_0}{r} \int_0^r r'(u_0 \pm u_{z,0} \frac{r'^2}{(r' + C)^2}) dr' \tag{13}$$

$$= \frac{en_0\mu_0}{r} \int_0^r u_0 r' \pm u_{z,0} \frac{r'^3}{(r' + C)^2} dr' \tag{14}$$

$$= \frac{en_0\mu_0}{r}\left(u_0 \frac{r^2}{2} \pm u_{z,0} \frac{f(r, C)}{2(r + C)}\right) \tag{15}$$

$$= \mu_0 en_0\left(u_0 \frac{r}{2} \pm u_{z,0} \frac{f(r, C)}{2r(r + C)}\right) \tag{16}$$

We can then write the bulk-flow core pressure as,

$$p_0^{(\pm)} = I_1 \pm I_2 \pm I_3 + I_4 \tag{17}$$

where,

$$I_1 = \frac{e^2 n_0^2 u_0^2 \mu_0}{2} \int_0^{r_p} r dr \tag{18}$$

$$I_2 = \frac{e^2 n_0^2 u_{z,0} u_0 \mu_0}{2} \int_0^{r_p} \frac{f(r, C)}{r(r + C)} dr \tag{19}$$

$$I_3 = \frac{e^2 n_0^2 u_0 u_{z,0} \mu_0}{2} \int_0^{r_p} \frac{r^3}{(r + C)^2} dr \tag{20}$$

$$I_4 = p_0 = \frac{e^2 n_0^2 u_{z,0}^2 \mu_0}{2} \int_0^{r_p} f(r, C) \frac{r}{(r + C)^3} dr \tag{21}$$

which is obtained from,

$$J_z^{(\pm)} B_\theta^{(\pm)} = e^2 n_0^2 \mu_0 \left(u_0 \pm u_{z,0} \frac{r^2}{(r+C)^2} \right) \left(\frac{u_0}{2} r \pm u_{z,0} \frac{f(r,C)}{2r(r+C)} \right) \tag{22}$$

The extra terms retained from treating this problem with a finite velocity on-axis introduces an imaginary component into the pressure, as well as a second-order polylogarithm. This is without considering the ramifications of treating the velocity constructed from the weak form of the problem as such a case is represented by the formidable,

$$u_z^{(0)}(r) = u_{z,0} \frac{(T_0 + T_1 r + T_2 r^2)^2}{(T(r) + C_B n_0 r^2)^2} \tag{23}$$

cubic, pureflow expression. Another area to investigate is that of self-similar solutions between this and the analytic cubic, pureflow current. Foregoing this for the moment we have for the core pressure of the cubic, bulk-flow vortex an expression whose complex modulus must be squared to evaluate its magnitude. Then, the square-root must be taken. The radicand is,

$$|p_0^{(\pm)}|^2 = p_0^{(\pm)} p_0^{(\pm)*} = \left(I_1 \pm I_2 \pm I_3 + I_4 \right) \left(I_1 \pm I_2^* \pm I_3 + I_4 \right) \tag{24}$$

Expanding this yields sixteen terms that must be evaluated for the computation of the radicand whose square-root must be computed. We can simplify them by writing,

$$I_2 = C_2(a + bi) \tag{25}$$

where,

$$\begin{aligned} a &= -C^2\pi^2 - 4Cr_p + \frac{r_p^2}{2} + 2C^2(1 + 3\ln\left(\frac{C}{r_p}\right)) \ln\left(\frac{C}{C+r_p}\right) + 6Li_2(\frac{C+r_p}{C}) \\ b &= -6C^2\pi \ln\left(\frac{C}{C+r_p}\right) \end{aligned} \tag{26}$$

What we will obtain is,

$$|p_0^{(\pm)}|^2 = I_1^2 \pm 2(I_1 I_3 + I_1 I_4 + I_3 I_4) + I_3^2 \tag{27}$$
$$\pm 2C_2 a(I_1 + I_4) + 2C_2 a I_3 + C_2^2(a^2 + b^2) \tag{28}$$