A Kadomtsev stable Shear-flow Stabilized Z-pinch Equilibrium

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A magnetohydrodynamic shear flow stabilized Z-pinch equilibrium may be constructed which also satisfies the ideal stability criterion against $m = 0$ "sausage" modes in a Z-pinch without sheared axial flow, $\frac{\mathrm{d}}{\mathrm{d}r} \ln\left(\frac{p}{\langle j_z \rangle^{\overline{\gamma}}}\right) \leq 0$, everywhere in the pinch and for all positive parameter values by exchanging the non-relativistic Bennett profile from number density to drift flow velocity, and then inserting an enhanced thermodynamic gradient based on ideal treatment of the Shumlak-Hartman shear-flow stabilization inequality. The drift flow velocity must be linked to the center of mass flow. The momentum equation provides the thermodynamic closure relating pressure, density, and temperature. An isotropic heat flux provides the fluid closure based on the enhanced thermodynamic gradient. Both the ideal shear-flow stabilization criterion, and a non-ideal criterion are strongly satisfied towards the axis in an arbitrarily small pinch radius. However, this shear-flow stabilized equilibrium fails the static $m = 1$ criterion as it is reversed instead, making a flowing spectral analysis, and higher-order study of the equilibrium necessary to further investigate the stabilizing effect of the shear, alongside the growth of compressible modes, and other important physics.

The non-relativistic Bennett profile[1,2],

$$n(r) = \frac{n_0}{(1+\Xi^2 r^2)^2} \tag{1}$$

$$\Xi^2 = \frac{C_B n_0}{T} \tag{2}$$

$$C_B = \frac{\mu_0 e^2 u_{z,0}^2}{16 k_B} \tag{3}$$

describes an approximately self-similar solution to a magnetically self-focusing stream with uniform, sub-luminal velocities possessed by the ions and electrons. Swapping this from number density to flow gives the profile,

$$U_z(r) = u_{z,0} \frac{T(r)^2}{(T(r) + C_B n_0 r^2)^2} \tag{4}$$

because the temperature of this system must now be non-uniform to compensate for the now uniform number density. If this axial drift flow velocity is treated for the moment as also being the stabilizing axial center of mass flow velocity,

$$U_z(r) = u_z(r) \tag{5}$$

then we can seek an answer from the Shumlak-Hartman[3] criterion,

$$\frac{\mathrm{d}u_z}{\mathrm{d}r} > 0.1 k V_A \tag{6}$$

$$k = \frac{2\pi}{L} \tag{7}$$

$$V_A = \frac{B_\theta}{\sqrt{\rho_0 \mu_0}} \tag{8}$$

as to what it would take for Eq. (4) to be shear-flow stabilized, i.e., to satisfy Eq. (6). The linkage of the drift flow and the center of mass flow in this way is not un-physical, and there are a number of cases in which it is possible. One being when the electron velocity is treated as zero, so that the ions carry the current which does not violate basic plasma physics. It is just a statement that the electron velocity is treated as being zero, e.g., if their equilibrium fluid displacement was uniform,

$$\vec{\xi} = \vec{\xi}_0 \tag{9}$$

so that the derivative of this would yield zero.

Considering then a power law temperature,

$$T(r) = \frac{T_p}{r_p^n} r^n \tag{10}$$

we will further treat Shumlak-Hartman in the long-wavelength limit, $L \to \infty$, because that gives $R_m = \sigma \mu u L \to \infty$ which corresponds to ideal magnetohydrodynamic behavior in the axial direction. This does not preclude resistive effects, e.g., magnetic diffusion, from mattering in other directions. These effects just do not influence the axial behavior of the resulting uniform, steady, stationary pure shear-flow stabilized[4–7] Z-pinch equilibrium that we will form from this process if there is a power law exponent, n, that is sufficient for the task.

In this long wavelength limit, provided the magnetic field does not run, we have,

$$u_z' > 0 \tag{11}$$

which for Eq. (4) becomes,

$$u_{z,0} 2 T C_B n_0 r \left(\frac{T' r - 2T}{(T + C_B n_0 r^2)^3}\right) > 0 \tag{12}$$

With Eq. (10) inserted this leads to,

$$n > 2 \tag{13}$$

as the necessary enhancement to the temperature gradient required for shear-flow stabilization. The denominator and prefix can be thrown away so long as $r = 0$ is ignored. It is acceptable to do this because the plasma state on the axis can

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be specified directly instead. Inserting the minimum integer, $n = 3$ which satisfies this criterion we have,

$$u_z(r) = u_{z,0}\frac{r^2}{(r+C)^2} \tag{14}$$

$$C = \frac{\mu_0 e^2}{16 k_B}\frac{n_0 u_{z,0}^2 r_p^3}{T_p} \tag{15}$$

Treating Eq. (14) as also the drift flow, which is not true in general, but also not un-physical because there are physical situations that we can treat where this is true, e.g., when the ions carry the current, then we can determine that this equilibrium,

$$n = n_0 \tag{16}$$
$$\vec{u} = u_z(r)\hat{e}_z \tag{17}$$
$$\vec{J} = e n_0 u_z(r)\hat{e}_z \tag{18}$$
$$\vec{B} = B_\theta \hat{e}_\theta \tag{19}$$
$$B_\theta = \frac{\mu_0}{r}\int_0^r r' J_z(r') dr' \tag{20}$$
$$p(r) = p_0 - \int_0^r J_z B_\theta dr \tag{21}$$
$$p_0 = \int_0^{r_p} J_z B_\theta dr \tag{22}$$

satisfies Ideal Magnetohydrodynamics (MHD). Mass conservation is trivialized by the uniform number density, and incompressible flow velocity. Momentum is satisfied because the uniform density, steady Z-pinch ansatz contains no inertial forces, and the magnetic field, and plasma pressure here both integrate. This force balance equation represents the thermodynamic closure to the system as it relates the pressure to the temperature and density. Energy is satisfied as well because of the axisymmetric nature of this system, the orthogonality of the MHD flow and magnetic field, and its steady character. The enhanced temperature gradient closes the system of equations by relating the temperature to an isotropic heat flux via a Fick's Law to satisfy power balance.

It must be said before moving on that this equilibrium does not obey the ideal gas law, and it does not need to as the relationship between temperature, pressure, and number density is contained in the force balance of the momentum equation. In fact, it can be said that it should not, because this equilibrium is built from a place of shear-flow stabilization. Imposing $p \propto n_0 T$ on such a system runs counter to the reality of shear-flow stabilized systems, which are non-ideal[8]. Normalized profiles of this equilibrium are shown in Figure (1).

$$\tilde{u}_z(\phi) = \frac{u_z(r)}{u_{z,0}} = \frac{\phi^2}{(\phi+1)^2} \tag{23}$$
$$\tilde{u}_z'(\phi) = u_z'(r)\frac{C}{2u_{z,0}} = \frac{\phi}{(\phi+1)^3} \tag{24}$$
$$\tilde{B}_\theta = B_\theta \frac{2}{(e n_0 u_{z,0})^2 \mu_0 C} \tag{25}$$
$$= \frac{\phi^3 - 3\phi^2 - 6\phi(1-\ln(\phi+1)) + 6\ln(\phi+1)}{\phi(\phi+1)} \tag{26}$$
$$\frac{dp}{d\phi} = -\frac{(e n_0 u_{z,0} C)^2 \mu_0}{2}\frac{\phi}{(\phi+1)^3}\Big(\phi^3 - 3\phi^2 \tag{27}$$
$$- 6\phi(1-\ln(\phi+1)) + 6\ln(\phi+1)\Big) \tag{28}$$
$$\rightarrow \frac{d\tilde{p}}{d\phi} = -\frac{\phi}{(\phi+1)^3}\Big(\phi^3 - 3\phi^2 \tag{29}$$
$$- 6\phi(1-\ln(\phi+1)) + 6\ln(\phi+1)\Big) \tag{30}$$

We will evaluate two limits of the normalized shear measure,

$$S^{(I)} = \frac{u_z'}{k V_A} = u_z'\frac{L}{2\pi}\frac{\sqrt{\rho\mu_0}}{B_\theta} \tag{31}$$

that Shumlak and Hartman originally considered. The limit of arbitrarily small plasma radius,

$$\lim_{r\to 0^+} S^{(I)} > 0.1 \tag{32}$$

and the limit of arbitrarily small pinch radius,

$$\lim_{r_p\to 0^+} S^{(I)} > 0.1 \tag{33}$$

Evaluating Equation (32) and Equation (33) here depends on evaluating the limits of the function,

$$S(r) = \frac{u_z'}{B_\theta} = \frac{4C}{e n_0 \mu_0}\frac{r^2}{(r+C)^2}\frac{1}{f(r,C)} \tag{34}$$

For the limit as plasma radius goes to zero $S(r)$ is indeterminate. Two applications of L'Hopital's rule are required to lift it yielding,

$$\lim_{r\to 0^+} S(r) = \frac{4C}{e n_0 \mu_0}\lim_{r\to 0^+}\frac{2}{(r+C)^2 f'' + 4(r+C)f' + 2f} \tag{35}$$

with,

$$f(r,C) = r^3 - 3r^2 C - 6rC^2(1+\ln\left(\frac{C}{r+C}\right)) - 6C^3\ln\left(\frac{C}{r+C}\right)$$

which diverges because $f(0^+,C) = f'(0^+,C) = f''(0^+,C) = 0$ while the numerator remains finite and non-zero. So, this equilibrium is shear-flow stabilized in an ideal sense for arbitrarily small plasma radius.

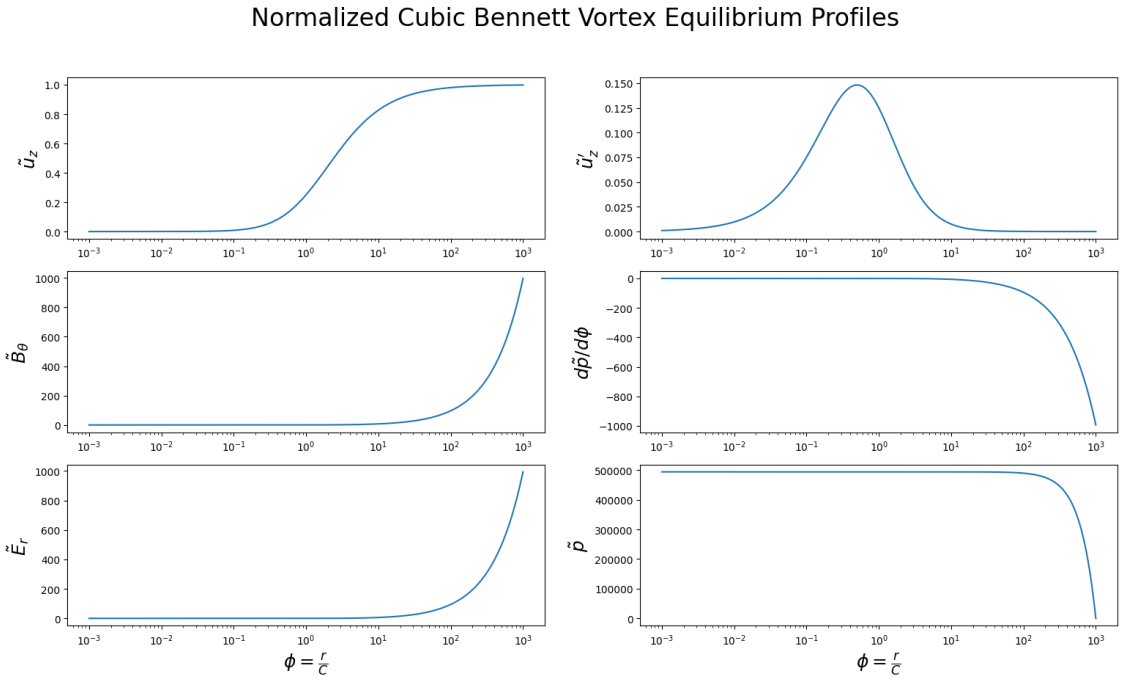


FIG. 1. Normalized radial profiles of the cubic pureflow Bennett–Shumlak–Hartman equilibrium. The dimensionless radial coordinate is $\phi = r/C$, with the pinch boundary located at $\phi_p = r_p/C$. The current density increases monotonically with radius and exceeds its enclosed-area average, $j_z > \langle j_z \rangle$, throughout the interior of the pinch. The normalizations and closed forms for the profiles are shown in Eq. (23) - (30)

For arbitrarily small pinch radius we have,

$$C = \tilde{C} r_p^3 \tag{36}$$

so that,

$$\lim_{r_p \to 0^+} S(r) = \frac{4}{e n_0 \mu_0} \tilde{C} r_p^3 \frac{r^2}{(r + \tilde{C} r_p^3)^2} \frac{1}{f(r, \tilde{C} r_p^3)} \tag{37}$$

We consider a scaling relationship between r, and r_p,

$$r = \alpha r_p \tag{38}$$

so that we have,

$$\lim_{r_p \to 0^+} S(r) = \frac{4}{e n_0 \mu_0} \tilde{C} \alpha^2 \lim_{r_p \to 0^+} \frac{r_p^3}{(\tilde{C} r_p^2 + \alpha)^2} \frac{1}{f(\alpha r_p, \tilde{C} r_p^3)} \tag{39}$$

$$= \frac{4}{e n_0 \mu_0} \tilde{C} \alpha^2 \lim_{r_p \to 0^+} \frac{r_p^3}{\alpha^2} \frac{1}{r^3} \tag{40}$$

$$= \frac{4}{e n_0 \mu_0} \frac{\tilde{C}}{\alpha^3} \tag{41}$$

Plugging this back into Equation (33) we arrive at the limiting condition,

$$20 L \frac{\sqrt{\rho \mu_0}}{\pi e n_0 \mu_0} \frac{\tilde{C}}{\alpha^3} > 1 \tag{42}$$

which is strongly satisfied towards the axis.

We will evaluate two limits of the normalized shear measure based on the non-ideal AWDG shear criterion,

$$S^{(NI)} = \frac{u'_z r_p}{\left| \frac{\mathrm{d}\ln(p)}{\mathrm{d}\ln(r)} \right|} \tag{43}$$

where the limit of arbitrarily small plasma radius becomes,

$$\lim_{r \to 0^+} S^{(NI)} > \frac{\omega_* L}{2\pi} \tag{44}$$

where ω_* is some characteristic frequency to make the dimensions match, and the limit of arbitrarily small pinch radius,

$$\lim_{r_p \to 0^+} S^{(NI)} > \frac{\omega_* L}{2\pi} \tag{45}$$

The construction of the non-ideal criterion here based on Angus et. al's work is made on the approximate recommendations presented there. A non-trivial vacuum pressure is now required,

$$p_0 = p_{vac} + \int_0^{r_p} J_z B_\theta dr \tag{46}$$

because the logarithmic pressure gradient becomes,

$$\frac{\mathrm{d}\ln(p)}{\mathrm{d}\ln(r)} = -r \frac{J_z(r) B_\theta(r)}{p(r)} \tag{47}$$

so the value will diverge to infinity at the pinch radius if $p_{vac} = 0$. It is acceptable to treat the amplitude of this as,

$$\left|\frac{\mathrm{d}\ln(p)}{\mathrm{d}\ln(r)}\right| = r\frac{J_z B_\theta}{p(r)} \tag{48}$$

because both the numerator, and denominator are positive for all non-trivial plasma radius, r, and shear localization, C. With $J_z(r) = en_0 u_z(r)$ then we have,

$$S^{(NI)} = \frac{2r_p C}{en_0}\frac{1}{r^2(r+C)}\frac{p(r)}{B_\theta(r)} \tag{49}$$

so that the limit in the context of an arbitrarily small plasma radius becomes,

$$\lim_{r\to 0^+} S^{(NI)} = \infty \tag{50}$$

meaning this equilibrium is shear-flow stabilized for an arbitrarily small plasma radius.

In the limit of arbitrarily small pinch radius we again fix the plasma radius at which we are observing the system as a fraction of the pinch radius so that it does not slide outside of the system bounds as we are taking the limit. Then, evaluating this we obtain,

$$x = \frac{r}{r_p} \tag{51}$$

$$\lim_{r_p\to 0^+} S^{(NI)}(xr_p, r_p) = \frac{1}{4k_B T_p}\frac{u_{z,0}}{n_0}\frac{p_{vac}}{x^4} \tag{52}$$

so that for fixed normalized radius we have a limiting condition which becomes strongly satisfied as we move towards the axis, i.e., $x \to 0^+$ which is valid in the context of the limit we took as we can always choose an r smaller than whatever r_p is as we contract the latter to maintain consistency.

In the absence of sheared axial flows the stabilizing effect of the compressional work by the magnetic field to the interchange mode dominates over the instability drive when the equilibrium pressure profile satisfies the Kadomtsev-equivalent condition[9]

$$\frac{\mathrm{d}}{\mathrm{d}r}\ln\left(\frac{p}{\langle j_z\rangle^\gamma}\right) \leq 0 \tag{53}$$

where $\langle j_z\rangle = \int j_z dA/\pi r^2$ is area-averaged current density[10]. Here that becomes,

$$\langle j_z\rangle = \frac{2en_0 u_{z,0}}{r^2}\int_0^r \frac{r'^3}{(r'+C)^2}dr' \tag{54}$$

and Eq. (53) leads to,

$$\frac{p'}{p} - \gamma\frac{\langle j_z\rangle'}{\langle j_z\rangle} \leq 0 \tag{55}$$

For the equilibrium shown here we can prove that this is satisfied for all physically admissible parameter values of the positive-current pureflow vortex. This follows from the structure of the current density, Eq. (18), which can be seen from Eq. (14) and Eq. (15) to be always positive for any combination of plasma parameters and positive flow speed modulus squared. Then, the same can be said for the magnetic field via Ampere's Law as integrating this always positive current density will result in a magnetic field that has the same property. Consequently, $p' = -J_z B_\theta < 0$ holds throughout the domain for any C, and $p > 0$ does as well because $p_0 > \int_0^r J_z B_\theta dr, \forall r\ s.t.\ 0 < r < r_p$. Similarly, $\langle j_z\rangle > 0$ holds likewise, as the integrand is the same as for B_θ. This leaves us with the need to prove that $\langle j_z\rangle' > 0$. Writing,

$$\langle j_z\rangle(r) = \frac{2}{r^2}F(r), \quad F(r) = \int_0^r j_z(r')r'dr' \tag{56}$$

we have,

$$\langle j_z\rangle' = \frac{2}{r}\left(j_z - \langle j_z\rangle\right) \tag{57}$$

The current density here increases monotonically with radius because,

$$j_z'(r) = 2en_0 u_{z,0} C\frac{r}{(r+C)^3} > 0 \tag{58}$$

so we have $j_z(s) < j_z(r)$ for any $s < r$ allowing us to write,

$$\int_0^r j_z(s)sds < \int_0^r j_z(r)sds \tag{59}$$

but $j_z(r)$ can be pulled outside the RHS of Eq. (59) to give,

$$\int_0^r j_z(s)sds < j_z(r)\frac{r^2}{2} \tag{60}$$

which leaves us with,

$$j_z(r) > \frac{2}{r^2}\int_0^r j_z(s)sds \tag{61}$$

and this is precisely,

$$j_z > \langle j_z\rangle \tag{62}$$

therefore,

$$\langle j_z\rangle' > 0, \quad 0 < r < r_p \tag{63}$$

so that we have Eq. (55) expressing the Kadomtsev-equivalent criterion satisfied.

The same paper that gives the Kadomtsev-equivalent $m = 0$ condition discusses the static stability criterion for kink stability,

$$\frac{\mathrm{d}(rp_B)}{\mathrm{d}r} \leq 0 \tag{64}$$

where,

$$p_B = \frac{B_\theta^2}{2\mu_0} \tag{65}$$

so that we have,

$$d(rp_B) = rj_zB_\theta - \frac{B_\theta^2}{2\mu_0} \tag{66}$$

We know that,

$$\langle j_z \rangle = \frac{2}{\mu_0 r} B_\theta \tag{67}$$

so that,

$$j_z > \frac{2}{\mu_0 r} B_\theta \tag{68}$$

and,

$$rj_zB_\theta > 2\frac{B_\theta^2}{\mu_0} \tag{69}$$

Substituting Eq. (69) into Eq. (66) we have,

$$\frac{\mathrm{d}(rp_B)}{\mathrm{d}r} > 2\frac{B_\theta^2}{\mu_0} - \frac{B_\theta^2}{2\mu_0} \tag{70}$$

which becomes,

$$\frac{\mathrm{d}(rp_B)}{\mathrm{d}r} > 3\frac{B_\theta^2}{2\mu_0} > 0 \tag{71}$$

While this shear-flow stabilized Z-pinch equilibrium built from the non-relativistic Bennett profile and the Shumlak-Hartman shear-flow stabilization criterion satsifies the unsheared $m = 0$ Kadomtsev-equivalent stability criterion, it reverses the static $m = 1$ criterion. This does not mean that this magnetohydrodynamic equilibrium is unstable to $m = 1$ modes, because it is not a static equilibrium. Rather, it is a flowing shear-flow stabilized Z-pinch equilibrium so the stabilizing effect of the shear here requires further study as these criteria are for unsheared, and static cases, respectively.

A shear-flow Z-pinch equilibrium is constructed from the non-relativistic Bennett profile, and the Shumlak-Hartman shear-flow stabilization criterion using an enhanced thermodynamic gradient. This flowing, shear-flow equilibrium is shown to satisfy a Kadomtsev equivalent criterion for unsheared $m = 0$ stability while reversing the satisfaction of a criterion for static $m = 1$ stability.

The magnetohydrodynamic equilibrium constructed here does not obey the ideal gas law, and it does not need to because it comes from a background of shear-flow stabilization. These systems are non-ideal in nature. The thermodynamic closure of this system is instead expressed in the momentum balance. That is where the relationship between density, temperature, and pressure is contained. The ideal gas law is an ad hoc closure that is not necessary in order to formulate ideal magnetohydrodynamics. What is necessary for ideal MHD is an isotropic pressure. This isotropic pressure, which is obtained from the force balance, is the relevant quantity for studying the confinement properties of this equilibrium, and not the ideal gas pressure.

Consequently, it is important for the next steps of this research to focus on a spectral analysis of this flowing equilibrium to identify and study the stabilizing effect of the shear flow on sausage and kink instabilities.

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