# Some Notes on Bennett-Shumlak Vorticity: Navier-Stokes & Beltrami Flow

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Consider the Navier-Stokes equation,

$$\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \nabla \vec{u} = -\frac{1}{\rho} \nabla p + \nu \nabla^2 \vec{u} \tag{1}$$

which models the momentum balance in an incompressible flow that contains an isotropic viscosity and pressure as well as a uniform density. An axisymmetric axial flow,

$$\vec{u} = u_z(r) \hat{z} \tag{2}$$

results in,

$$\frac{\nu}{r} \frac{\mathrm{d}}{\mathrm{d}r} \left( r \frac{\mathrm{d}u_z}{\mathrm{d}r} \right) = \frac{1}{\rho} \frac{\mathrm{d}p}{\mathrm{d}z} \tag{3}$$

where $p$ is the fluid pressure. Given a form for the axial flow this can be calculated. For example, if the flow were that of a cubic, pureflow Bennett vortex,

$$u_z(r) = u_{z,0} \frac{r^2}{(r + C_{B,T})^2} \tag{4}$$

then the pressure would be,

$$p(r, z) = 2\rho\nu u_{z,0} C_{B,T} z \left[ \frac{2C_{B,T} - r}{(r + C_{B,T})^4} \right] \tag{5}$$

The consistency of this form requires that,

$$\frac{\partial p}{\partial r} = 0 \tag{6}$$

in order for the NS equations to remain satisfied for this axisymmetric flow.

$$\frac{\partial p}{\partial r} = 2\rho\nu C_{B,T} u_{z,0} z \frac{\partial}{\partial r} \left( \frac{2C_{B,T} - r}{(r + C_{B,T})^4} \right) \tag{7}$$

with the derivative being,

$$\frac{\partial}{\partial r}\left(\frac{2C_{B,T}-r}{(r+C_{B,T})^4}\right) = \Big(-(r+C_{B,T})^{-4} - 4(2C_{B,T}-r)(r+C_{B,T})^{-5}\Big) \tag{8}$$

$$= -\left(\frac{r+C_{B,T}}{(r+C_{B,T})^5} + \frac{4(2C_{B,T}-r)}{(r+C_{B,T})^5}\right) \tag{9}$$

$$= -\left(\frac{r+C_{B,T}+8C_{B,T}-4r}{(r+C_{B,T})^5}\right) \tag{10}$$

$$= -\left(\frac{9C_{B,T}-3r}{(r+C_{B,T})^5}\right) \tag{11}$$

yielding,

$$\frac{\partial p}{\partial r} = 0 \tag{12}$$

on the loci,

$$\implies z = 0 \text{ OR } r = 3C_{B,T} \tag{13}$$

so that a cubic-pureflow Bennett-Shumlak vortex is an exact solution to NS on these loci within this formulation. This steady axial axisymmetric flow contains no convective nonlinearities, so NS reduces to a force balance between the fluid pressure gradients and viscous forces related to the diffusion of velocity. The main challenge in this regime is ensuring consistency amongst the pressure gradient forces and this leads to the aforementioned constraint.

It is noteworthy that this function represents a local solution to Navier-Stokes, nonetheless, as there is nothing guaranteeing this fact. For example, consider Beltrami flow,

$$\nabla \times \vec{u} = \lambda_\omega \vec{u} \tag{14}$$

which is analogous to the idea of Taylor states in magnetohydrodynamics. Is it possible for a Beltrami system to possess a Bennett-Shumlak character? Meaning, we search for solutions to Equation (14) of the form,

$$\vec{u} = \vec{u}_{BF} + u_{BS}(r)\hat{z} = \vec{u}_{BF} + u_{z,0}\frac{r^2}{(r+C_{B,T})^2}\hat{z} \tag{15}$$

Superficially, there are seven options for the spatial dependence that the added character can possess: $\{r, (r,\theta), (r,z), (r,\theta,z), (\theta,z), \theta, z\}$. Beltrami flow in cylindrical coordinates, and with a Bennett-Shumlak axial flow, requires,

$$u_r = \frac{1}{\lambda_\omega}\left(\frac{1}{r}\frac{\partial u_z}{\partial \theta} - \frac{\partial u_\theta}{\partial z}\right) \tag{16}$$

$$u_\theta = \frac{1}{\lambda_\omega}\left(\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}\right) \tag{17}$$

$$u_z = \frac{1}{\lambda_\omega}\left(\frac{1}{r}\frac{\partial (r u_{\theta)}}{\partial r} - \frac{1}{r}\frac{\partial u_r}{\partial \theta}\right) = u_{z,0}\frac{r^2}{(r+C_{B,T})^2} = u_{BS}(r) \tag{18}$$

This demands that the radial flow either be independent of azimuth, or uniformly dependent on it. Similarly, the radial flow also must be independent of axial coordinate in order to maintain consistency with our demand for Bennett-Shumlak flow. Therefore, we can write,

$$\frac{\partial u_r}{\partial \theta} = g(r) \tag{19}$$

and using the latter of these observations note that,

$$u_\theta = -\frac{1}{\lambda_\omega}\frac{\partial u_z}{\partial r} \tag{20}$$

Inserting these into the equation which determines the axial flow we have,

$$u_z(r) = \frac{1}{\lambda_\omega}\left(\frac{1}{r}\frac{\partial(ru_\theta)}{\partial r} - \frac{1}{r}g(r)\right) \tag{21}$$

$$= \frac{1}{\lambda_\omega}\left[\frac{1}{r}\frac{\partial}{\partial r}\left( - r\frac{1}{\lambda_\omega}\frac{\partial u_z}{\partial r}\right) - \frac{1}{r}g(r)\right] \tag{22}$$

$$= -\frac{1}{\lambda_\omega^2}\left[\frac{1}{r}\left(\frac{\partial u_z}{\partial r} + ru_z''\right) + \frac{\lambda_\omega}{r}g(r)\right] \tag{23}$$

Obtaining $g(r)$ is then straightforward,

$$g(r) = -\lambda_\omega r u_z(r) - \frac{1}{\lambda_\omega}(u_z' + ru_z'') \tag{24}$$

Turning to the Beltrami equation for the radial flow component we encounter a problem,

$$u_r = \frac{1}{\lambda_\omega}\left(\frac{1}{r}\frac{\partial u_z}{\partial \theta} - \frac{\partial u_\theta}{\partial z}\right) \tag{25}$$

$$= -\frac{1}{\lambda_\omega}\frac{\partial u_\theta}{\partial z} \tag{26}$$

as this needs to be zero otherwise the radial component will have an axial dependence which violates our earlier assumption that it didn't. We could still attempt to integrate the above and produce a form for the swirl but we would notice that it is inconsistent with our earlier requirement for its spatial dependence. Consequently, we can conclude that the Beltrami flow system studied here is incompatible with a Bennett-Shumlak character. This is noteworthy because it demonstrates that there are fluid regimes which Bennett-Shumlak vortices are inconsistent with such as constant-$\lambda_\omega$ Beltrami embeddings.