Cubic Pureflow Bennett-Angus Vortices

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Abstract

The non-ideal Angus shear-flow stabilization criterion is studied in the context of the Bennett nonlinearity. When the thermal ion estimate is used the exact same power of enhanced thermodynamic gradient is found to be required for shear-flow stabilization as in the case of the ideal Shumlak-Hartman criterion, and therefore the cubic, pureflow Bennett vortex remains the minimum thermal energy shear-flow stabilized state. When the non-ideal Angus shear-flow stabilization criterion is considered then the cubic, pureflow Bennett vortex is found to be shear-flow stabilized for both arbitrarily small plasma radius, and arbitrarily small pinch radius.

Thermal Ion Bennett Shear

Let us consider the consequences of exceeding the estimate for the upper bound of the required shear to achieve shear-flow stabilization[1] based on thermal ion velocity. Our inequality of interest then reads,

$$\frac{\mathrm{d}u_z}{\mathrm{d}r} \geq \frac{v_{t,i}}{r_p}$$

where the thermal ion velocity is,

$$v_{t,i} = \sqrt{\frac{2k_B T_i}{m_i}}$$

We are interested in the shear of the following axial flow profile,

$$u_z(r) = u_{z,0}\frac{T^2(r)}{(T(r) + C_B n_0 r^2)^2} \tag{1}$$

with

$$C_B = \frac{\mu_0 e^2 u_{z,0}^2}{16 k_B}$$

which represents the non-relativistic Bennett nonlinearity [2] exchanged completely from density to flow. The shear of Equation (1) is given by,

$$\frac{u_z'}{u_{z,0}} = 2TT'(T + C_B n_0 r^2)^{-2} - 2T^2(T' + 2C_B n_0 r)(T + C_B n_0 r^2)^{-3}$$

which can be simplified by pulling out a common factor,

$$\begin{aligned}\frac{u_z'}{u_{z,0}} &= 2T(T + C_B n_0 r^2)^{-2}\left(T' - T(T' + 2C_B n_0 r)(T + C_B n_0 r^2)^{-1}\right) \\ &= 2T(T + C_B n_0 r^2)^{-2}\frac{T' C_B n_0 r^2 - 2T C_B n_0 r}{T + C_B n_0 r^2} \\ &= 2T(T + C_B n_0 r^2)^{-2} C_B n_0 r \frac{T' r - 2T}{T + C_B n_0 r^2}\end{aligned} \tag{2}$$

To work more clearly with this thermal ion shear-flow criterion we square both sides so that it becomes,

$$u_z'^2 > \frac{2k_B T_i(r)}{m_i r_p^2}$$

Observe that the above can be written as,

$$\frac{1}{2}m_i u_z'^2 > \frac{k_B}{r_p^2} T_i(r)$$

Squaring u_z' yields,

$$u_z'^2 = 4u_{z,0}^2 T^2 (C_B n_0 r)^2 \left(\frac{T' r - 2T}{(T + C_B n_0 r^2)^3}\right)^2$$

where,

$$T(r) = T_e(r) + T_i(r)$$

Continuing, we have,

$$\begin{aligned} 2\frac{m_i r_p^2 u_{z,0}^2}{k_B}(C_B n_0 r)^2 T_i(r)\left(\frac{T' r - 2T}{(T + C_B n_0 r^2)^3}\right)^2 &\geq 1 \\ \therefore 2\frac{m_i r_p^2 u_{z,0}^2}{k_B}(C_B n_0 r)^2 T_i(r)\left(\frac{T' r - 2T}{(T + C_B n_0 r^2)^3}\right)^2 - 1 &\geq 0 \\ \implies 2\frac{m_i r_p^2 u_{z,0}^2}{k_B}(C_B n_0 r)^2 T_i(r)\left(\frac{T' r - 2T}{(T + C_B n_0 r^2)^3}\right)^2 - \frac{(T + C_B n_0 r^2)^6}{(T + C_B n_0 r^2)^6} &\geq 0 \end{aligned}$$

This last expression has a common denominator so we can throw this away into the zero on the RHS provided it does not equal zero itself. For the power-law temperatures we will be considering this only occurs at $r = 0$, but to maintain a well-posed problem we can just specify the state of the system here. That leaves us with,

$$2\frac{m_i r_p^2 u_{z,0}^2}{k_B}(C_B n_0 r)^2 T_i \left(T' r - 2T\right)^2 - (T + C_B n_0 r^2)^6 \geq 0 \tag{3}$$

choosing a power law temperature profile,

$$T(r) = C_T r^n = C_{T,e} r^n + C_{T,i} r^n = \left(\frac{T_{p,e}}{r_{p,e}^n} + \frac{T_{p,i}}{r_{p,i}^n}\right) r^n$$

We must next consider how the pinch radius of the electron and ion plasma and edge plasma temperatures are ordered. For our purposes it is simplest if we orient our approach to this by considering what happens when either the temperatures or pinch radius are coincident, and what are the ways in which we can have only one temperature. When the temperatures are coincident, and the pinch radii are coincident then one profile is sufficient and we have,

$$C_T = 2\frac{T_p}{r_p^n}$$

otherwise we will need to consider regimes where there is a large difference between one of the quantities so that the temperature profile is dominated by one species. However, even so the coefficient above will still remain multiplied by a factor of order unity. Choosing then,

$$T_e = T_i = \frac{1}{2}T$$

we have Equation (3) become,

$$\frac{m_i r_p^2 u_{z,0}^2}{k_B}(C_B n_0 r)^2 T\left(T'r - 2T\right)^2 - (T + C_B n_0 r^2)^6 \geq 0$$

and then inserting, $T(r) = C_T r^n$, we have,

$$A(C_B n_0 r)^2 C_T r^n \left(C_T n r^n - 2C_T r^n\right)^2 - (C_T r^n + C_B n_0 r^2)^6 \geq 0$$

which becomes,

$$A(C_B n_0)^2 C_T^3 r^{3n+2}\left(n - 2\right)^2 - (C_T r^n)^6 (1 + C r^{2-n})^6 \geq 0 \tag{4}$$

where,

$$C = \frac{C_B n_0}{C_T}$$
$$A = \frac{m_i r_p^2 u_{z,0}^2}{k_B}$$

Dividing a factor of C_T^5 into both sides, which is only invalid if the pinch radius is infinite, or the edge temperature is zero, then we have,

$$AC^2 r^{3n+2}(n-2)^2 - C_T r^{6n}(1 + C r^{2-n})^6 \geq 0 \tag{5}$$

Consider the consequences of taking $n = 2$ in Equation (5). The inequality then becomes,

$$-C_T^{(2)} r^{12}(1 + C)^6 \geq 0$$

where,

$$C_T^{(2)} = \frac{T_p}{r_p^2}$$

but this is impossible to satisfy because the negative sign multiplies a positive quantity to yield a negative one, which can never be greater than zero, and only equal when $r = 0$ which is outside the domain of study, so an enhancement of $n = 2$ is not sufficient to yield shear-flow stabilization.

Now, consider the consequences of taking $n = 3$, which is the next level of exponent power. Then, we have from Equation (5),

$$\begin{aligned} &AC^2 r^{11} - C_T^{(3)} r^{18} (1 + Cr^{-1})^6 \geq 0 \\ &\implies AC^2 r^{11} - C_T^{(3)} r^{12} (r + C)^6 \geq 0 \end{aligned}$$

which ends up as,

$$\begin{aligned} r(r+C)^6 &\leq \frac{AC^2}{C_T^{(3)}} \\ &\lesssim 10^{-44} \frac{m_i n_0^2}{k_B T_p^3} u_{z,0}^6 r_p^{11} \end{aligned} \tag{6}$$

with,

$$C = \frac{C_B n_0}{C_T^{(3)}} \simeq 10^{-22} n_0 u_{z,0}^2 r_p^3 T_p^{-1} \text{ [m]}$$

While this is not unrestrictive, we can still see that this criterion is satisfied for small plasma radius because $\lim_{r \to 0} LHS \to 0$ and the RHS remains finite and non-zero because it is independent of radius. Therefore, this thermal ion shear criterion requires the exact same enhancement to the thermodynamic gradient to be shear-flow stabilized as the ideal Shumlak-Hartman criterion[3]. Meaning, the $n = 3$ cubic, pureflow Bennett vortex is the minimum energy shear-flow stabilized state when treating this family in the context of this thermal ion shear flow criterion.

Summary and Conclusion - Thermal Ion Shear Criterion

We have considered the requirement for the Bennett profile to produce a shear-flow stabilized Z-pinch when the shear threshold that must be exceeded is written based on the quotient between ion thermal speed and pinch radius as considered previously by researchers studying the non-ideality of shear-flow stabilized Z-pinches. The resulting minimum energy state is the exact same as it was previously, and the cubic, pureflow Bennett vortex remains shear-flow stabilized for arbitrarily small plasma radius. This accords with the investigation of this family in the context of the ideal Shumlak-Hartman shear-flow criterion.

Bennett-Angus Criterion

The non-ideal shear-flow criterion found by Angus, van de Wetering, Dorf, and Geyko is,

$$u_z' \geq \frac{C}{(kr_p)^\alpha} \gamma_0 (kr_p)$$

where $\alpha \approx 1$, $C \sim 1$, and the shear-less growth rate $\gamma_0(kr_p)$ is related to q_P, where,

$$q_P \approx \left|\frac{\mathrm{d}\ln(p)}{\mathrm{d}\ln(r)}\right|$$

this logarithmic pressure logarithmic gradient is related to the nominal gradient via the chain rule, yielding,

$$\frac{\mathrm{d}\ln(p)}{\mathrm{d}\ln(r)} = \frac{\mathrm{d}\ln(p)}{\mathrm{d}r} / \frac{\mathrm{d}\ln(r)}{\mathrm{d}r} = -rB_\theta J_z \frac{1}{p}$$

for a pure Z-pinch. The above can be simplified further in this case to rest entirely on the shoulders of the axial plasma current density via Ampere's Law,

$$\begin{aligned}\frac{\mathrm{d}\ln(p)}{\mathrm{d}\ln(r)} &= -rJ_z(r)B_\theta(r)\frac{1}{p(r)} \\ &= -\mu_0 J_z(r)\frac{\int_0^r r' J_z(r')dr'}{p_0 - \mu_0 \int_0^r \frac{J_z(r')}{r'} \int_0^{r'} r'' J_z(r'')dr''dr'}\end{aligned}$$

Here we have used,

$$\begin{aligned}\frac{\mathrm{d}p}{\mathrm{d}r} &= -J_z B_\theta \\ rB_\theta &= \mu_0 \int_0^r r' J_z(r')dr' \\ p_0 &= p_{vac} + \int_0^{r_p} J_z B_\theta dr\end{aligned}$$

The axial current density we are interested in studying is of course,

$$J_z = en_0 u_{z,0} \frac{T(r)^2}{(T(r) + C_B n_0 r^2)^2}$$

which is the Bennett nonlinearity exchanged from number density to flow. There are as many options for temperature profile here as there are axisymmetric temperature profiles which can be represented. In this work we will choose a power-law temperature profile,

$$\begin{aligned}T(r) &= C_T^{(n)} r^n \\ C_T^{(n)} &= \frac{T_p}{r_p^n}\end{aligned}$$

because in that case the Bennett current density integrates to produce an analytic equilibrium azimuthal magnetic field. For a general power exponent, n, we have the current density,

$$J_z(r) = en_0 u_{z,0} \frac{1}{(1 + Cr^{2-n})^2}$$

Immediately, we can observe that for $n = 2$, the shear structure disappears. For $n = 3$ it has been shown elsewhere that this is the minimum thermal energy state at which this profile is shear-flow stabilized according to the ideal criterion. The first section of this note showed this is true as well when using the thermal ion shear threshold.

Now, we will study the $n = 3$ case. For the Angus criterion then we have,

$$u_z' \gtrsim \frac{L}{2\pi r_p}\left|\frac{\mathrm{d}\ln(p)}{\mathrm{d}\ln(r)}\right|$$

Immediately, we can observe that the approach of infinite magnetic Reynold's number, R_m, will not work as it does in the case of Shumlak-Hartman. It would be incorrect to attempt the same asymptotic elimination based on an infinite magnetic Reynold's number in the radial direction as well because infinitely large pinch radius will also cause the shear structure to collapse on account of the consequential disappearance of the enhanced thermodynamic gradient. This is all naturally understood because the Angus criterion is non-ideal. The shear on the LHS is given in Equation (2), and the magnitude of the logarithmic pressure logarithmic gradient can be investigated for its tractability in the cubic case.

This tractability necessitates that two integrals be analytic,

$$I_1(r) = \int_0^r r' J_z(r') dr' \tag{7}$$

$$I_2(r) = \int_0^r \frac{1}{r'} J_z(r') I_1(r') dr' \tag{8}$$

Fortunately, in the cubic case, both integrals are. The first reads,

$$I_1(r) = en_0 u_{z,0} \int_0^r \frac{r'^3}{(r'+C)^2} dr' = en_0 u_{z,0} \frac{f(r,C)}{2(r+C)}$$

where,

$$f(r,C) = r^3 - 3r^2C - 6rC^2\left(1 + \ln\left(\frac{C}{r+C}\right)\right) - 6C^3 \ln\left(\frac{C}{r+C}\right) \tag{9}$$

so that I_2 reads,

$$\begin{aligned} I_2(r) &= \frac{(en_0 u_{z,0})^2}{2} \int_0^r \frac{r' f(r',C)}{(r'+C)^3} dr' \\ &= \frac{(en_0 u_{z,0})^2}{4} \frac{f_2(r,C)}{(r+C)^2} \end{aligned}$$

with,

$$f_2(r,C) = r(r+2C)(r^2 - 12rC - 15C^2) + 6C^2(r+C)\ln\left(\frac{C}{r+C}\right)(-5C - 3r + (r+C)\ln\left(\frac{C}{r+C}\right))$$

which leaves the aforementioned logarithmic value as,

$$\frac{\mathrm{d}\ln(p)}{\mathrm{d}\ln(r)} = -\mu_0(en_0u_{z,0})^2\frac{r^2}{r+C}\frac{2f(r,C)}{4p_0(r+C)^2-\mu_0(en_0u_{z,0})^2f_2(r,C)} \tag{10}$$

In the cubic case, the shear on the LHS becomes,

$$u_z' = 2u_{z,0}C\frac{r}{(r+C)^3}$$

Turning our attention to the RHS we have a magnitude that must be taken. The main question is whether it is possible to accomplish this by just dropping the negative sign in Equation (10). This is only possible if the rest of the expression is positive-valued.

Fortunately, we can see for $r > 0$ and $C > 0$, we have

$$\left|\frac{\mathrm{d}\ln(p)}{\mathrm{d}\ln(r)}\right| = \mu_0(en_0u_{z,0})^2\frac{r^2}{r+C}\left|\frac{2f(r,C)}{4p_0(r+C)^2-\mu_0(en_0u_{z,0})^2f_2(r,C)}\right|$$

and the question of whether we can just drop the negative sign in representing this magnitude, which formerly is the square root of a complex modulus squared, rests on whether $f(r,C)$ is a positive-definite function, taken here to mean,

$$f(0,C) = 0$$
$$f(r,C) > 0, \forall r, C > 0$$

and whether

$$D(r,C) = 4p_0(r+C)^2 - \mu_0(en_0u_{z,0})^2f_2(r,C) > 0$$

for $r > 0$, $C > 0$.

Lemma 1: $f(r,C) > 0 \quad \forall r, C > 0$

This can be shown by the structure of the non-dimensionalized forms derivatives. The non-dimensionalization of $f(r,C)$ can be accomplished as shown below with the dimensionless parameter, $\phi = r/C$ utilized,

$$\tilde{f}(\phi) = \frac{f(\phi)}{C^3} = \phi^3 - 3\phi^2 - 6\phi(1+\ln\left(\frac{1}{1+\phi}\right)) - 6\ln\left(\frac{1}{1+\phi}\right) \tag{11}$$

$$= \phi^3 - 3\phi^2 - 6\phi\left(1-\ln(1+\phi)\right) + 6\ln(\phi+1) \tag{12}$$

$$= \phi^3 - 3\phi^2 - 6\phi + 6(\phi+1)\ln(\phi+1) \tag{13}$$

For $\tilde{f}(\phi = 0)$, which correspnds to $r = 0$, we have $\tilde{f}(0) = 0$ by inspection as the polynomial parts become trivialized, and the only non-zero aspect becomes trivialized logarithmically. Since $\tilde{f}(\phi = 0) = 0$, it is sufficient for $\tilde{f}(\phi) > 0\,\forall\phi > 0$ if $\tilde{f}'(\phi) > 0$ throughout this domain. The first derivative of $\tilde{f}$ is,

$$\begin{aligned}\frac{\mathrm{d}\tilde{f}}{\mathrm{d}\phi} &= 3\phi^2 - 6\phi - 6 + 6\left(\ln(\phi+1) + \frac{\phi+1}{\phi+1}\right)\\ &= 3\phi^2 - 6\phi + 6\ln(\phi+1)\\ &= 3(\phi^2 - 2\phi + 2\ln(\phi+1))\end{aligned}$$

It can be verified by inspection that $\tilde{f}'(0) = 0$. The derivative of this is,

$$\begin{aligned}\frac{\mathrm{d}^2\tilde{f}}{\mathrm{d}\phi^2} &= 6\phi - 6 + \frac{6}{\phi+1} = 6\left(\phi - 1 + \frac{1}{\phi+1}\right)\\ &= 6\left(\frac{(\phi-1)(\phi+1)+1}{\phi+1}\right)\\ &= 6\frac{\phi^2}{\phi+1}\\ &> 0, \quad \forall\phi > 0\end{aligned}$$

Therefore we know that $\tilde{f}'$ has the same property since it begins at zero, and its rate of change only increases throughout the domain. Since this is true, it is also true that $\tilde{f}$ has the same property as well, i.e., that it is positive definite, for the same reasons.

Lemma 2: $D(r, C) > 0, \quad 0 \leq r \leq r_p$

Is the statement,

$$4p_0(r+C)^2 - \mu_0(en_0u_{z,0}^2)f_2(r,C) > 0, \quad 0 \leq r \leq r_p$$

true? This is the central question. If the above holds then we can say,

$$\left|\frac{\mathrm{d}\ln(p)}{\mathrm{d}\ln(r)}\right| = \mu_0(en_0u_{z,0})^2\frac{r^2}{r+C}\frac{2f(r,C)}{4p_0(r+C)^2 - \mu_0(en_0u_{z,0})^2f_2(r,C)}$$

and investigate directly whether the cubic, pureflow Bennett vortex is shear-flow stabilized according to Angus. The denominator here is made up of just the plasma pressure which we know can be written,

$$\begin{aligned}p(r) &= p_0 - \int_0^r J_zB_\theta dr'\\ &= p_{vac} + \int_0^{r_p} J_zB_\theta dr - \int_0^r J_zB_\theta dr'\\ &= p_{vac} + \int_r^{r_p} J_zB_\theta dr'\end{aligned}$$

with $p_{vac} \geq 0$. Here, the product of current density and magnetic field is non-negative so that integrating their product across the given span will yield a positive value that will be added to the vacuum pressure to give the plasma pressure. The rest of the denominator comes from the product,

$$\begin{aligned}J_zB_\theta &= A\frac{r^2}{(r+C)^2}\frac{f(r,C)}{2r(r+C)} \\ &= \frac{A}{2}\frac{r}{(r+C)^3}f(r,C)\end{aligned}$$

The denominator just becomes,

$$D(r,C) = 4(r+C)^2p(r)$$

which satisfies,

$$D(r,C) > 0$$

because of the non-negative nature of the product J_zB_θ, and the requirement that $p_{vac} > 0$. If $p_{vac} = 0$, then we have a singularity occur at $r = r_p$ because $D(r_p, C, p_{vac} = 0) = 0$.

Turning back to the Angus criterion then we have,

$$u_z' \geq \frac{L}{2\pi r_p}A\frac{r^2}{r+C}\frac{2f(r,C)}{4p_0(r+C)^2 - Af_2(r,C)}$$

which becomes,

$$\begin{aligned}2u_{z,0}C\frac{r}{(r+C)^3} &\geq \frac{L}{2\pi r_p}A\frac{r^2}{r+C}\frac{2f(r,C)}{4p_0(r+C)^2 - Af_2(r,C)} \\ \implies 2\pi r_p u_{z,0}\frac{C}{LA} &\geq \frac{rf(r,C)(r+C)^2}{4p_0(r+C)^2 - Af_2(r,C)}\end{aligned}$$

It can be seen from Equation (9) for $f(r,C)$, that $\lim_{r\to 0^+} f(r,C) = 0$, and since $D(0,C) > 0$, we have for an arbitrarily small plasma radius,

$$2\pi r_p u_{z,0}\frac{C}{LA} \geq 0$$

indicating that the cubic, pureflow Bennett vortex is also shear-flow stabilized for arbitrarily small plasma radius according to the Angus criterion. When the pinch radius is taken to be arbitrarily small instead we have a problem that collapses strictly at $r_p = 0$ so we must evaluate this limit approaching from the right instead, yielding,

$$0 \geq \lim_{r_p\to 0^+} r(r+C)^2\frac{f(r,C)}{4p_{vac}(r+C)^2 - Af_2(r,C)}$$

which is indeterminate on the RHS. However, this can be evaluated, for example by CAS, or sufficient application of L'Hopitals rule, to yield,

$$0 \geq \lim_{r_p\to 0^+}\frac{r^4}{4p_{vac} - Ar^2}$$

then we can finish evaluating the RHS by considering the plasma radius as being a fixed fraction of the pinch radius,

$$r = \xi r_p, 0 \leq \xi \leq 1$$

and we will find that we obtain,

$$\begin{aligned} 0 &\geq \lim_{r \to 0^+} \frac{\xi^4 r_p^4}{4p_{vac} - A\xi^2 r_p^2} \\ &\geq 0 \\ p_{vac} &> 0 \end{aligned}$$

Indicating the satisfaction of this shear-flow stabilization property extends here to arbitrarily small pinch radius as well.

Summary & Conclusion

Both the thermal ion shear estimate, and the non-ideal Angus criterion for shear-flow stabilization were studied in the context of Bennett vorticity. The cubic pureflow Bennett vortex was found to remain the minimum thermal energy state in the thermal ion shear threshold, and the non-ideal Angus criterion was found to be satisfied for arbitrarily small plasma and pinch radius. This satisfaction rests on the assumption that

$$\begin{aligned} \left|\frac{\mathrm{d}\ln(p)}{\mathrm{d}\ln(r)}\right| &= -\frac{\mathrm{d}\ln(p)}{\mathrm{d}\ln(r)} \\ &= \mu_0(en_0u_{z,0})^2 \frac{r^2}{r+C} \frac{2f(r,C)}{4p_0(r+C)^2 - \mu_0(en_0u_{z,0})^2 f_2(r,C)} \end{aligned}$$

which is satisfied because of the structure of the numerator and denominator as detailed in the analysis.

Consequentially, we find in the previous section that the cubic, pureflow Bennett vortex is shear-flow stabilized for arbitrarily small plasma and pinch radius provided $p_{vac} > 0$. If $p_{vac} = 0$ then we will have a divergence occur in the RHS of the Angus criterion at the pinch radius.

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