The Relativistic Electron Current of a Shear-Flow Stabilized Cubic, Pureflow Bennett-Shumlak-Hartman Vortex

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This brief note derives the relativistic electron flow necessary to satisfy the shear-flow stabilized cubic, pureflow Bennett-Shumlak-Hartman vortex plasma current density. It is shown to be invariant to radial length contraction. The lack of inertial forces in the Z-pinch ansatz avoids the appearance of relativistic γ in the two-fluid system so that the same Ideal MHD force balance is obtained as in the non-relativistic case. Self-similarity between the relativistic electron plasma current and the non-relativistic current allows for the backing out of the necessary relativistic electron flow. The relativistically-modified Shumlak-Hartman criterion is investigated and found to be satisfied.

The two-fluid equations for a collisionless, hydrogen plasma with electron inertia retained,

$$\frac{\partial n_e}{\partial t} + \nabla \cdot (n_e \vec{u}_e) = 0 \tag{1}$$

$$\frac{\partial n_i}{\partial t} + \nabla \cdot (n_i \vec{u}_i) = 0 \tag{2}$$

$$m_e n_e \left(\frac{\partial}{\partial t} + \vec{u}_e \cdot \nabla \right) \vec{u}_e = -e n_e (\vec{E} + \vec{u}_e \times \vec{B}) - \nabla p_e \tag{3}$$

$$m_i n_i \left(\frac{\partial}{\partial t} + \vec{u}_i \cdot \nabla \right) \vec{u}_i = e n_i (\vec{E} + \vec{u}_i \times \vec{B}) - \nabla p_i \tag{4}$$

$$\frac{3}{2} n_e \left(\frac{\partial}{\partial t} + \vec{u}_e \cdot \nabla \right) T_e + p_e \nabla \cdot \vec{u}_e + \nabla \cdot \vec{q}_e = S_e \tag{5}$$

$$\frac{3}{2} n_i \left(\frac{\partial}{\partial t} + \vec{u}_i \cdot \nabla \right) T_i + p_i \nabla \cdot \vec{u}_i + \nabla \cdot \vec{q}_i = S_i \tag{6}$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{7}$$

$$\nabla \times \vec{B} = \mu_0 e (n_i \vec{u}_i - n_e \vec{u}_e) \tag{8}$$

$$n_i - n_e = 0 \tag{9}$$

$$\nabla \cdot \vec{B} = 0 \tag{10}$$

reduce to,

$$\vec{J} \times \vec{B} = \nabla p \tag{11}$$

$$\nabla \cdot \vec{q} = S \tag{12}$$

when an axial, axisymmetric ansatz is taken for the electron and ion flows,

$$\vec{u}_i(r) = u_i(r) \hat{z} \tag{13}$$

$$\vec{u}_e(r) = u_e(r) \hat{z} \tag{14}$$

$$\tag{15}$$

This is true even if *ad hoc* relativistic modifications, e.g.,

$$m_s = \gamma(r) m_{s,0} \tag{16}$$

$$n_s = \gamma(r) n_{s,0} \tag{17}$$

are made because the Z-pinch ansatz with a uniform density results in no inertial forces,

$$\hat{r} \cdot (\vec{u} \cdot \nabla) \vec{u} = u_r \frac{\partial u_r}{\partial r} + \frac{u_\theta}{r} \frac{\partial u_r}{\partial \theta} + u_z \frac{\partial u_r}{\partial z} - \frac{u_\theta^2}{r} \tag{18}$$

$$\hat{\theta} \cdot (\vec{u} \cdot \nabla) \vec{u} = u_r \frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r} \frac{\partial u_\theta}{\partial \theta} + u_z \frac{\partial u_\theta}{\partial z} + \frac{u_\theta u_r}{r} \tag{19}$$

$$\hat{z} \cdot (\vec{u} \cdot \nabla) \vec{u} = u_r \frac{\partial u_z}{\partial r} + \frac{u_\theta}{r} \frac{\partial u_z}{\partial \theta} + u_z \frac{\partial u_z}{\partial z} \tag{20}$$

To link the plasma current density,

$$\vec{J} = en(\vec{u}_i - \vec{u}_e) = en(u_i(r) - u_e(r)) \hat{z} \tag{21}$$

with the MHD flow,

$$\vec{u} = \frac{m_i \vec{u}_i + m_e \vec{u}_e}{m_i + m_e} \tag{22}$$

that matters for the Shumlak-Hartman shear-flow stabilization criterion then stationary ions can be selected,

$$\vec{u}_i \approx 0 \tag{23}$$

which requires relativistic electrons to avoid the pre-factor,

$$\vec{u} = \frac{m_e}{m_i + m_e} \vec{u}_e \tag{24}$$

The question then becomes, what is the relativistic electron flow that corresponds to the plasma current density which is integrable when,

$$\vec{U} = \vec{u}_i - \vec{u}_e = u_{z,0} \frac{r^2}{(r + C_{B,T})^2} \hat{z} \tag{25}$$

We can answer this question via self-similarity,

$$J_z = -e n_0 \gamma(r) u_e(r) = e n_0 u_z(r) \tag{26}$$

where,

$$\gamma(r) = \frac{1}{\sqrt{1 - \frac{u_e^2}{c^2}}} \tag{27}$$

Note that the existence of a negative sign, or lack thereof, in the integrable plasma current density does not matter because

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we square both sides to eliminate the square root in the denominator.

$$\gamma^2(r)u_e(r)^2 = u_z^2(r) \tag{28}$$

$$\therefore \frac{u_e^2}{1-\frac{u_e^2}{c^2}} = u_z^2 \tag{29}$$

$$\implies c^2u_e^2 = (c^2-u_e^2)u_z^2 \tag{30}$$

$$\implies u_e^2c^2 + u_e^2u_z^2 = c^2u_z^2 \tag{31}$$

$$\therefore u_e^2 = \frac{c^2u_z^2}{c^2+u_z^2} \tag{32}$$

Note that the axial flow velocity, u_z, is invariant to relativistic length contraction,

$$u_z(r) = u_{z,0}\frac{(r/\gamma)^2}{(r/\gamma + C_{B,T}/\gamma)^2} \tag{33}$$

$$= u_{z,0}\frac{\gamma^2}{\gamma^2}\frac{r^2}{(r+C_{B,T})^2} \tag{34}$$

$$= u_{z,0}\frac{r^2}{(r+C_{B,T})^2} \tag{35}$$

so that the self-similar relativistic electron plasma current, Equation (32 will be as well.

The shear of this profile experiences a relativistic modification,

$$\frac{\mathrm{d}u}{\mathrm{d}r} \simeq \frac{\mathrm{d}u_e}{\mathrm{d}r} = c\frac{\mathrm{d}\big(u_z(c^2+u_z^2)^{-1/2}\big)}{\mathrm{d}r} \tag{36}$$

$$= c\left[u_z'(c^2+u_z^2)^{-1/2} - u_z^2u_z'(c^2+u_z^2)^{-3/2}\right] \tag{37}$$

$$= \frac{\mathrm{d}u_z}{\mathrm{d}r}c\left[(c^2+u_z^2)^{-1/2} - u_z^2(c^2+u_z^2)^{-3/2}\right] \tag{38}$$

$$= \frac{\mathrm{d}u_z}{\mathrm{d}r}c\left[\frac{c^2+u_z^2-u_z^2}{(c^2+u_z^2)^{3/2}}\right] \tag{39}$$

$$= u_z'c^3/(c^2+u_z^2)^{3/2} \tag{40}$$

Does the above satisfy the relativistic Shumlak-Hartman shear-flow stabilization criterion? The RHS of this becomes,

$$0.1kV_A = \frac{2\pi}{10L}\frac{B_\theta}{\sqrt{\rho\mu_0}} \tag{41}$$

$$= \frac{\pi}{5L}\frac{B_\theta}{\sqrt{\rho_0\mu_0}}\frac{1}{\gamma} \tag{42}$$

$$= \frac{\pi}{5L}\frac{B_\theta}{\sqrt{\mu_0\rho_0}}\sqrt{1-\frac{u^2}{c^2}} \tag{43}$$

so that in totality we evaluate,

$$\therefore \frac{\mathrm{d}u}{\mathrm{d}r} \simeq \frac{\mathrm{d}u_e}{\mathrm{d}r} = u_z'c^3/(c^2+u_z^2)^{3/2} \tag{44}$$

$$> \frac{\pi}{5L}\frac{B_\theta}{\sqrt{\rho_0\mu_0}}\sqrt{1-u^2/c^2} \tag{45}$$

We can move the radical term to the LHS,

$$\implies u_z'\frac{c^4}{(c^2+u_z^2)^{3/2}}\frac{1}{\sqrt{c^2-u^2}} > 0.1kV_A\Big|_{NR} \tag{46}$$

with infinite Reynold's magnetic number we can throw away the RHS of this as previously done when $L \to \infty$ and $B_\theta \not\to \infty$ which it does not for the plasma current density that we are exploring.

Simplifying the above, with,

$$\sqrt{c^2-u^2} = \sqrt{c^2 - \frac{c^2u_z^2}{c^2+u_z^2}} \tag{47}$$

$$= c\sqrt{\frac{c^2+u_z^2-u_z^2}{c^2+u_z^2}} \tag{48}$$

$$= c\sqrt{\frac{c^2}{c^2+u_z^2}} \tag{49}$$

$$= c^2\frac{1}{\sqrt{c^2+u_z^2}} \tag{50}$$

then we have on the LHS of Equation (46),

$$u_z'\frac{c^4}{(c^2+u_z^2)^{3/2}}\frac{(c^2+u_z^2)^{1/2}}{c^2} = u_z'\frac{c^2}{c^2+u_z^2} \tag{51}$$

$$\therefore u_z'\frac{c^2}{c^2+u_z^2} > 0.1kV_A\Big|NR \tag{52}$$

We do not actually even need to throw away the term on the right with our previous argument for doing so. Instead we simply take the limit as the plasma radius goes to zero. This kills the term on the right because of the structure of the magnetic field as we have previously computed,

$$\therefore \lim_{r\to 0} u_z'\frac{c^2}{c^2+u_z^2} = \lim_{r\to 0} u_z'\frac{c^2}{c^2} \tag{53}$$

$$= \frac{c^2}{c^2}\lim_{r\to 0} u_z' \tag{54}$$

$$= \lim_{r\to 0} u_{z,0}\frac{2rC_{B,T}}{(r+C_{B,T})^3} \tag{55}$$

$$= u_{z,0} * 0 \tag{56}$$

$$= 0 \tag{57}$$

Therefore we find that the relativistic Shumlak-Hartman shear-flow stabilization criterion for this flow is satisfied in arbitrarily small spaces weakly by equality. QED.

Let us also consider what happens when we take the simplification of infinite R_m via infinite pinch length, such as in a tokamak. Then we have,

$$u_z'\frac{c^4}{(c^2+u_z^2)^{3/2}}\frac{1}{\sqrt{c^2-u^2}} > 0 \tag{58}$$

which becomes, as before,

$$u_z'\frac{c^2}{c^2+u_z^2} > 0 \tag{59}$$

$$\implies LHS = u_{z,0}\frac{2rC_{B,T}}{(r+C_{B,T})^3}\frac{c^2}{c^2+(u_{z,0}\frac{r^2}{(r+C_{B,T})^2})^2} \tag{60}$$

$$= u_{z,0}\frac{2rC_{B,T}(r+C_{B,T})^4}{(r+C_{B,T})^3}\frac{c^2}{c^2(r+C_{B,T})^4+u_{z,0}^2r^4} \tag{61}$$

$$= u_{z,0}\frac{2rC_{B,T}c^2(r+C_{B,T})}{c^2(r+C_{B,T})^4+u_{z,0}^2r^4} \tag{62}$$

$$> 0 \tag{63}$$

At $r=0$ the above is well-defined, and satisfies the weak form, i.e., the criterion when equality with the threshold shear is achieved. The only singularities that could appear to possibly exist are those when the plasma radius is zero, and when the problem is trivial through $C_{B,T}=0$. Investigation of that topic is outside the scope of this note, however, and imply connections to $T_p \to \infty$.

Instead, we can throw away the whole denominator, because outside of this pathological case everything in this form will be non-zero. We can also throw away the $r+C_{B,T}$ factor in the numerator because we are not studying the ultraviolet-type pathology branch. Then, we are finally left with,

$$u_{z,0}C_{B,T}r > 0 \tag{64}$$

which will be satisfied for all non-trivial cubic, pureflow, Bennett-Shumlak-Hartman vortices. QED.