Cubic Pureflow Bennett Vortices: An Analytic Shear-flow Stabilized Z-pinch Equilibrium and Application to the MAST Edge Pedestal

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A shear-flow stabilized Z-pinch equilibrium is derived by exchanging the non-relativistic Bennett profile from number density to two-fluid plasma drift velocity. This drift velocity can be connected to the MHD flow that underlies the Shumlak-Hartman shear-flow stabilization criterion when the problem is studied from the ion rest frame, the electron rest frame, when the ions are treated as immobile relative to the electrons, or vice-versa. An electron-dominated two-fluid system is studied, and an axial, axisymmetric ansatz is required for the species velocities in the two-fluid theory in order to reduce to the MHD system. An enhanced thermodynamic gradient is required to achieve shear-flow stabilization. The cubic temperature profile gives the minimum thermal power state, and these vortices accurately solve the current density of the MAST edge pedestal across a parameter sweep.

The non-relativistic Bennett profile[1] is given by,

$$n(r) = n_0 \frac{1}{(1+\xi^2 r^2)^2} \tag{1}$$

where,

$$\xi^2 = \frac{b}{T} \tag{2}$$

and,

$$b = \frac{\mu_0 e^2 u_{z,0}^2}{8 k_B (T_e + T_i)} \tag{3}$$

Let us exchange this with a uniform flow that Bennett considered[2], and treat the resulting profile as the MHD flow of a Z-pinch,

$$\vec{u} = u_z(r)\hat{z} = \frac{u_{z,0}}{(1+\xi^2 r^2)^2}\hat{z} \tag{4}$$

in order to study what would be required for this profile to satisfy the Shumlak-Hartman criterion[3],

$$\frac{\mathrm{d}u_z}{\mathrm{d}r} > 0.1 k V_A \tag{5}$$

and achieve shear-flow stabilization. The MHD regime being considered here is entangled with the idea that the Reynold's magnetic number is infinite,

$$R_m = \sigma \mu u L \tag{6}$$

so that the Shumlak-Hartman shear-flow stabilization criterion in this regime becomes,

$$\frac{\mathrm{d}u_z}{\mathrm{d}r} > 0.1 k V_A \tag{7}$$

$$> \frac{1}{10}\frac{2\pi}{L}\frac{B}{\sqrt{\rho\mu_0}} \tag{8}$$

$$> 0 \tag{9}$$

provided the magnetic field obtained by treating this MHD flow as also supporting the plasma current density of a hydrogen plasma,

$$\vec{J} = en(\vec{u}_i - \vec{u}_e) \tag{10}$$

does not run to infinity. The simplification in the shear-flow stabilization criterion is obtained by the consideration that $L \to \infty$.

In order to treat the MHD flow in this manner,

$$\vec{u} = \frac{m_i \vec{u}_i + m_e \vec{u}_e}{m_i + m_e} \tag{11}$$

it is necessary to study the system from either the rest frame of the ions, the rest frame of the electrons, as if the ions were immobile, or as if the electrons were. In the first case we have,

$$\vec{u} = \frac{m_e}{m_i + m_e}\vec{u}_e^{(R)} \tag{12}$$

where the electron flow velocity observed in the ion rest frame, i.e., taking $\vec{u}_i \to 0$ in the above, is given by,

$$\frac{m_e}{m_e + m_i}\vec{u}_e^{(R)} = \vec{u} \tag{13}$$

$$= \frac{m_i \vec{u}_i + m_e \vec{u}_e}{m_i + m_e} \tag{14}$$

so that,

$$\vec{u}_e^{(R)} = \vec{u}_e + \frac{m_i}{m_e}\vec{u}_i \tag{15}$$

if the ions are taken to be immobile then the MHD flow, and plasma current density are linked together by the electron flow. Another possibility satisfying this is when the electrons are sufficiently relativistic that,

$$m_e(\gamma) >> m_i \tag{16}$$

then these electrons will provide the plasma current density from the perspective of the ion rest frame of reference, or a two-fluid regime where we consider the ion motion to be negligible in comparison to the electron motion,

$$u_e >> u_i \tag{17}$$

$$\therefore \vec{u}_i \approx 0 \tag{18}$$

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In this electron-dominated two-fluid case the plasma current density is given by,

$$\vec{J} = -en\vec{u}_e \tag{19}$$

and the MHD flow velocity is given by,

$$\vec{u} = \frac{m_e}{m_i + m_e}\vec{u}_e \tag{20}$$

For an electron flow that is relativistic against a non-relativistic ion background then we will have no effective mass factor show up in the plasma current density.

We can arrive at the MHD framework for a Z-pinch from the two-fluid equations for a collisionless hydrogen plasma with the electron inertia retained, and with mobile ions,

$$\frac{\partial n_e}{\partial t} + \nabla\cdot(n_e\vec{u}_e) = 0 \tag{21}$$

$$\frac{\partial n_i}{\partial t} + \nabla\cdot(n_i\vec{u}_i) = 0 \tag{22}$$

$$m_e n_e\left(\frac{\partial}{\partial t} + \vec{u}_e\cdot\nabla\right)\vec{u}_e = -en_e(\vec{E} + \vec{u}_e\times\vec{B}) - \nabla p_e \tag{23}$$

$$m_i n_i\left(\frac{\partial}{\partial t} + \vec{u}_i\cdot\nabla\right)\vec{u}_i = en_i(\vec{E} + \vec{u}_i\times\vec{B}) - \nabla p_i \tag{24}$$

$$\frac{3}{2}n_e\left(\frac{\partial}{\partial t} + \vec{u}_e\cdot\nabla\right)T_e + p_e\nabla\cdot\vec{u}_e + \nabla\cdot\vec{q}_e = S_e \tag{25}$$

$$\frac{3}{2}n_i\left(\frac{\partial}{\partial t} + \vec{u}_i\cdot\nabla\right)T_i + p_i\nabla\cdot\vec{u}_i + \nabla\cdot\vec{q}_i = S_i \tag{26}$$

$$\nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \tag{27}$$

$$\nabla\times\vec{B} = \mu_0 e(n_i\vec{u}_i - n_e\vec{u}_e) \tag{28}$$

$$n_i - n_e = 0 \tag{29}$$

$$\nabla\cdot\vec{B} = 0 \tag{30}$$

provided we take the ansatz,

$$\vec{u}_e = u_e(r)\hat{z} \tag{31}$$

$$\vec{u}_i = u_i(r)\hat{z} \tag{32}$$

Equations (21) and (22) represent the continuity of the system. For a uniform density $n_i = n_e = n_0$ they reduce to,

$$n_0\nabla\cdot(\vec{u}_i - \vec{u}_e) = 0 \tag{33}$$

which is satisfied for the given ansatz. Similarly, the momentum equations, Equations (23) and (24) can be added together to produce the MHD momentum balance,

$$\vec{J}\times\vec{B} = \nabla p \tag{34}$$

$$p = p_e + p_i \tag{35}$$

because the ansatz taken here eliminates the convective non-linearities on the LHS which represent the inertial forces on the plasma species, alongside the unsteady Eulerian component that disappears as well. The disappearance of the convective nonlinearities can be seen by observing that all components will be zero for an axial, axisymmetric flow, i.e. a Z-pinch,

$$\hat{r}\cdot(\vec{u}\cdot\nabla)\vec{u} = u_r\frac{\partial u_r}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_r}{\partial\theta} + u_z\frac{\partial u_r}{\partial z} - \frac{u_\theta^2}{r} \tag{36}$$

$$\hat{\theta}\cdot(\vec{u}\cdot\nabla)\vec{u} = u_r\frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_\theta}{\partial\theta} + u_z\frac{\partial u_\theta}{\partial z} + \frac{u_\theta u_r}{r} \tag{37}$$

$$\hat{z}\cdot(\vec{u}\cdot\nabla)\vec{u} = u_r\frac{\partial u_z}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_z}{\partial\theta} + u_z\frac{\partial u_z}{\partial z} \tag{38}$$

in general this will not be true, and even for an MHD flow with a Z-pinch component it is possible for the ion and electron flows to possess axisymmetric cross-terms that are scaled in a manner which cancels them out in the MHD velocity while introducing inertial forces onto the LHS.

Applying the ansatz to the energy equations, Equations (25) and (26) we find that the convective, and divergence terms disappear. Adding together the remainder we have an expression relating the thermal power density of the flow to the divergence of the heat flux,

$$\nabla\cdot\vec{q} = S \tag{39}$$

This depends on the specific form of the non-uniform temperature profile considered here, and can be integrated to produce the thermal power of the equilibrium.

The remaining aspects of the two-fluid system express Maxwell's equations and quasineutrality. Faraday's Law, Equation (27) becomes,

$$\nabla\times\vec{E} = 0 \tag{40}$$

requiring,

$$\frac{1}{r}\frac{\partial E_z}{\partial\theta} - \frac{\partial E_\theta}{\partial z} = 0 \tag{41}$$

$$\frac{\partial E_r}{\partial z} - \frac{\partial E_z}{\partial r} = 0 \tag{42}$$

$$\frac{1}{r}\frac{\partial(rE_\theta)}{\partial r} - \frac{1}{r}\frac{\partial E_r}{\partial\theta} = 0 \tag{43}$$

An axisymmetric electric field only introduces concerns about axial and azimuthal electric fields. Our assumption of electrons carrying the current does not contradict MHD even though typically MHD treats the ions as carrying the current, because we have shown that the two-fluid system for this dual axial, axisymmetric ansatz reduces to the MHD force balance.

With the electrons carrying the plasma current density, and for the axial, axisymmetric ansatz here, then the MHD force balance reduces to,

$$\vec{J}\times\vec{B} = -J_z B_\theta\hat{r} = \frac{\mathrm{d}p}{\mathrm{d}r}\hat{r} \tag{44}$$

because we know from Ampere's Law that the magnetic field is azimuthal and given by,

$$\mu_0 J_z = \frac{1}{r}\frac{\mathrm{d}(rB_\theta)}{\mathrm{d}r} \tag{45}$$

In this regime the above becomes,

$$-\mu_0 enu_e(r) = \frac{1}{r}\frac{\mathrm{d}(rB_\theta)}{\mathrm{d}r} \tag{46}$$

and this reduces to,

$$-\mu_0 enu_z(r) = \frac{1}{r}\frac{\mathrm{d}(rB_\theta)}{\mathrm{d}r} \tag{47}$$

when the electrons are suitably relativistic so that $m_e >> m_i$. This does not involve the introduction of any Lorentz factors because in this regime Equation (20) goes as $\sim \frac{m_e}{m_e}$ so that $\vec{u} \simeq \vec{u}_e$.

Turning back to the Shumlak-Hartman criterion after showing that the two-fluid equations for our ansatz, and electron-dominated regime, reduce to a standard Z-pinch equilibrium with a thermal power density that defines a thermal lifetime on which the confinement properties of this Z-pinch hold, the first order of business is to determine what makes the Bennett MHD flow,

$$u_z = \frac{u_{z,0}}{(1+\xi^2 r^2)^2} = \frac{u_{z,0}}{(1+\frac{C_B n_0}{T(r)}r^2)^2} \tag{48}$$

$$= u_{z,0}\frac{T(r)^2}{(T(r)+C_B n_0 r^2)^2} \tag{49}$$

shear-flow stabilized. The nonuniform temperature is necessary in order for the plasma pressure gradient to be non-trivial given that the number density is now uniform.

The shear of Equation (49) is,

$$\frac{\mathrm{d}u_z}{\mathrm{d}r} = u_{z,0}\frac{\mathrm{d}(T^2(T+C_B n_0 r^2)^{-2})}{\mathrm{d}r} \tag{50}$$

$$= u_{z,0}(2TT'(T+C_B n_0 r^2)^{-2} - 2T^2(T'+2C_B n_0 r)(T+C_B n_0 r^2)^{-3}) \tag{51}$$

$$= u_{z,0}\frac{C_B n_0 r}{(T+C_B n_0 r^2)^3}\left(2TT'r - 4T^2\right) \tag{52}$$

$$> 0 \tag{53}$$

An issue arises if the first term produces $0/0$. This can happen if the temperature is given by a power-law profile, and $r = 0$. To address this we will confine our study away from the origin because what is going on here can be specified by the boundary conditions to ensure the problem remains well-posed.

So long as $r \neq 0$ we have the first term disappear into the 0 on the RHS of the conditional. The structure of the temperature profile that leads to shear-flow stabilization then depends on the term in parentheses,

$$2TT'r - 4T^2 > 0 \tag{54}$$

We can simplify this into the form,

$$T'r - 2T > 0 \tag{55}$$

and integrate across the conditional to obtain the "weak" form of the shear-flow condition for this profile,

$$\frac{\mathrm{d}T}{\mathrm{d}r}r > 2T \tag{56}$$

$$\therefore \int \frac{1}{T}dT > 2\int \frac{1}{r}dr \tag{57}$$

$$\implies \ln(T) > 2\ln(r) \tag{58}$$

$$\implies T > r^2 \tag{59}$$

which is suggestive of the inverse parabolic temperature profiles seen in experimental shear-flow stabilized Z-pinches, e.g. Zap-HD[4].

A stronger approach is to consider a power-law temperature profile,

$$T = C_T r^n = \frac{T_p}{r_p^n} r^n \tag{60}$$

then we have a requirement for the power-law exponent,

$$rnC_T r^{n-1} - 2C_T r^n > 0 \tag{61}$$

$$\implies n > 2 \tag{62}$$

This suggests choosing $n = 3$, and evaluating the magnetic field that results from this choice. This vortex is named here to be the cubic, pureflow Bennett-Shumlak vortex because of the necessary enhanced thermodynamic gradient that is represented by the cubic temperature profile, the exchange between number density, and flow which placed the Bennett nonlinearity entirely onto the flow, and the Shumlak-Hartman shear-flow stabilization criterion that led to its construction.

Beyond evaluating the magnetic field, we must also verify that this profile is indeed a shear-flow stabilized Z-pinch by checking the Shumlak-Hartman criterion after obtaining a form for the magnetic field. First, let us see if we can integrate Equation (47) for the cubic, pureflow Bennett-Shumlak vortex,

$$u_z(r) = u_{z,0}\frac{T^2}{(T+C_B n_0 r^2)^2} \tag{63}$$

$$= u_{z,0}\frac{C_T^2 r^6}{(C_T r^3 + C_B n_0 r^2)^2} \tag{64}$$

$$= u_{z,0}\frac{C_T^2}{C_T^2}\frac{r^4}{r^4}\frac{r^2}{(r+\frac{C_B n_0}{C_T})^2} \tag{65}$$

$$= u_{z,0}\frac{r^2}{(r+C_{B,T})^2} \tag{66}$$

where,

$$C_{B,T} = (1.45967 * 10^{-22})\frac{n_0 u_{z,0}^2 r_p^3}{T_p}\ [m] \tag{67}$$

As it turns out, we can integrate this equation with the above via u-substitution or computer algebra system (CAS), e.g., Wolfram Mathematica. The resulting magnetic field is,

$$\vec{B} = B_\theta(r)\hat{\theta} \tag{68}$$

$$= -\frac{f(r,C_{B,T})}{2r(r+C_{B,T})}\hat{\theta} \tag{69}$$

where,

$$f(r,C_{B,T}) = f_1 + f_2 + f_3 + f_4 \tag{70}$$
$$f_1(r) = r^3 \tag{71}$$
$$f_2(r,C_{B,T}) = -3r^2 C_{B,T} \tag{72}$$
$$f_3(r,C_{B,T}) = -6rC_{B,T}^2 (1 + \ln\left(\frac{C_{B,T}}{r+C_{B,T}}\right)) \tag{73}$$
$$f_4(r,C_{B,T}) = -6C_{B,T}^3 \ln\left(\frac{C_{B,T}}{r+C_{B,T}}\right) \tag{74}$$

However, in order to produce a full equilibrium we must be able to integrate the momentum balance as well, Equation (44).

$$p(r) = p_0 - \int_0^r J_z B_\theta dr' \tag{75}$$
$$p(r_p) = 0 \implies p_0 = \int_0^{r_p} J_z B_\theta dr \tag{76}$$
$$\tag{77}$$

Despite its appearance this integral is tractable as it splits into four pieces,

$$\int_0^r \frac{r'}{(r'+C_{B,T})^3}(f_1+f_2+f_3+f_4)dr' \tag{78}$$
$$= P_1 + P_2 + P_3 + P_4 \tag{79}$$

$$P_1 = \int_0^r \frac{r'^4}{(r'+C_{B,T})^3} dr' \tag{80}$$
$$P_2 = -3C_{B,T}\int_0^r \frac{r'^3}{(r'+C_{B,T})^3} dr' \tag{81}$$
$$P_3 = -6C_{B,T}^2\int_0^r \frac{r'^2}{(r'+C_{B,T})^3}\left(1+\ln\left(\frac{C_{B,T}}{r'+C_{B,T}}\right)\right) dr' \tag{82}$$
$$P_4 = -6C_{B,T}^3\int_0^r \frac{r'}{(r'+C_{B,T})^3}\ln\left(\frac{C_{B,T}}{r'+C_{B,T}}\right) dr' \tag{83}$$

which are either rational functions amenable to u-substitution or the product of a rational function and a logarithmic term that can be integrated by parts after appropriate substitution.

Before applying this equilibrium to experimental plasma structures to see what sense it makes in reality, we must discuss how to solve for it. Note that a core flow speed can be specified without changing the shear to make this a "bulk" profile,

$$u_z^{\pm}(r) = u_0 \pm u_{z,0}\frac{r^2}{(r+C_{B,T})^2} \tag{84}$$

In the cubic, pureflow case we have,

$$u_{edge} = u_z(r_p) = u_{z,0}\frac{r_p^2}{(r_p+C_{B,T})^2} \tag{85}$$

The primary wrinkle in the above is the existence of an additional quadratic factor of $u_{z,0}$ inside of the shear layer placement $C_{B,T}$. If this value is sufficiently small, meaning,

$$C_{B,T} << r_p \tag{86}$$

then the flow constant root becomes degenerate,

$$u_{edge} \simeq u_{z,0} \tag{87}$$

Otherwise the boundary condition must be translated into an algebraic system,

$$u_{edge}(r_p + C_{B,T})^2 - u_{z,0}r_p^2 = 0 \tag{88}$$
$$\therefore u_{edge}(r_p^2 + 2r_pC_{B,T} + C_{B,T}^2) - u_{z,0}r_p^2 = 0 \tag{89}$$
$$\implies Au_{edge}u_{z,0}^4 + 2Bu_{edge}u_{z,0}^2 - r_p^2u_{z,0} + u_{edge}r_p^2 = 0 \tag{90}$$

where

$$A = \frac{\mu_0^2 e^4 n_0^2 r_p^6}{(16k_BT_p)^2} \tag{91}$$
$$B = \frac{\mu_0 n_0 e^2 r_p^4}{16k_BT_p} \tag{92}$$

that can technically be solved as all quartic equations can to give four different solutions for $u_{z,0}$. Nothing requires these roots to be real, and a pattern of complex roots shows up in the results presented in the article. The existence of complex roots relates to the structure of these quartic solutions which are based on closed form expression obtained in the 1800s by the Italian mathematician Ruffini[5]. An in-depth exploration of this topic is beyond the scope of this supplement.

In the bulk case we have,

$$u_{edge} = u_0 \pm u_{z,0}\frac{r_p^2}{(r_p+C_{B,T})^2} \tag{93}$$
$$\therefore u_{edge} - u_0 = \pm u_{z,0}\frac{r_p^2}{(r_p+C_{B,T})^2} \tag{94}$$

so that the structure of the algebraic system remains largely unchanged except for the substitution implied by the above. Of course, a subtlety is introduced by this, for if $u_{edge} = u_0$, then only trivial solutions to the flow roots exist. Evidently, within this equilibrium family it is inadmissible for there to be a symmetry between the edge and core flows, because such a uniformity would cause the shear to collapse between the span of the column as a consequence.

Let us display what this equilibrium looks like. The plasma pressure involves logarithmic terms that introduce numerical oscillations into a finite precision representation of the structure. These logarithmic terms also can show up elsewhere, e.g., when calculating drifts of this equilibrium due to their presence in the magnetic field. Rather than use the explicit form of the magnetic field, or plasma pressure, it is cleaner to integrate them numerically starting from the flow profile.

The flow, and shear, both depend on the $C_{B,T}$ parameter and the solutions to the flow speed roots. We consider a layer-normalized, or "core" basis, which is called this because it is small when the plasma radius, r, is relative to the location of the shear layer that is characterized by $C_{B,T}$. This latter point will be shown momentarily.

$$\phi = \frac{r}{C_{B,T}} \tag{95}$$

so that the normalized flow can be written as,

$$u_z(r) = u_{z,0}\frac{r^2}{(r+C_{B,T})^2} \tag{96}$$

$$= u_{z,0}\frac{r^2}{C_{B,T}^2}\frac{1}{(\phi+1)^2} \tag{97}$$

$$\therefore \tilde{u_z}(\phi) = \frac{u_z(r)}{u_{z,0}} = \frac{\phi^2}{(\phi+1)^2} \tag{98}$$

For $C_{B,T} >> r \to \phi << 1$ we have a profile that is small, and parabolic near the origin. For $r >> C_{B,T} \to \phi >> 1$ we have a profile that saturates at the asymptotic limit of the flow speed. This profile is the flow of a Z-pinch that terminates at some pinch radius so in reality we cannot use any values which are outside of,

$$\phi_p = \frac{r_p}{C_{B,T}} \tag{99}$$

Where this value is located depends on studying a specific configuration.

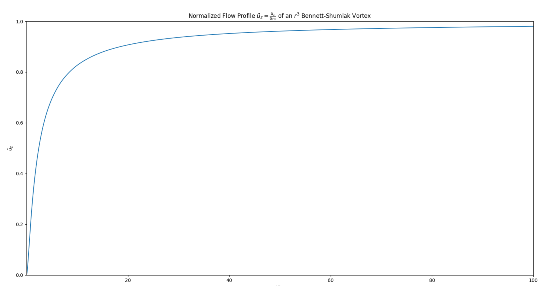

FIG. 1. Normalized cubic, pureflow Bennett-Shumlak vortex profile. The range over which the "core" variable, ϕ, varies in this this plot is $\phi \in [10^{-6}, 10^2]$ as that gives a view at the asymptotic structure of the profile as $r \to r >> C_{B,T}$.

The normalized shear can be obtained similarly, starting from,

$$\frac{\mathrm{d}u_z}{\mathrm{d}r} = u_{z,0}\frac{\mathrm{d}}{\mathrm{d}r^2(r+C_{B,T})^{-2}} \tag{100}$$

$$= u_{z,0}\left(2r(r+C_{B,T})^{-2} - 2r^2(r+C_{B,T})^{-3}\right) \tag{101}$$

$$= u_{z,0}\left(\frac{2r(r+C_{B,T})}{(r+C_{B,T})^3} - \frac{2r^2}{(r+C_{B,T})^3}\right) \tag{102}$$

$$= u_{z,0}\left(\frac{2rC_{B,T}}{(r+C_{B,T})^3}\right) \tag{103}$$

Which we can write with respect to the "core" variable,

$$\frac{\mathrm{d}u_z}{\mathrm{d}r} = u_{z,0}\frac{2}{C_{B,T}}\frac{\phi}{(\phi+1)^3} \tag{104}$$

Equation (103) evidently requires a different normalization than the strict flow normalization of the flow profile,

$$\tilde{u}'_z = u'_z\frac{C_{B,T}}{u_{z,0}} \tag{105}$$

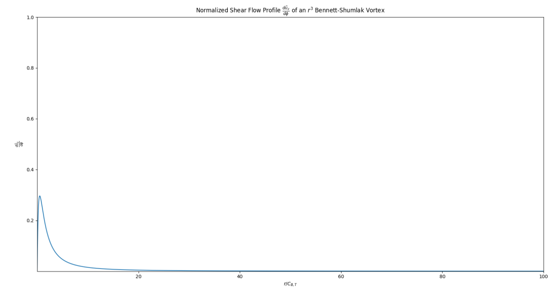

FIG. 2. Normalized cubic, pureflow Bennett-Shumlak vortex shear profile. The range over which the core variable, ϕ, varies is the same as in Figure (1). The shear is shown to asymptotically approach zero for long values of plasma radius, and the extremal value is located close to the pinch axis.

The factor of 2 is chosen to stay in the normalized expression in order to give the normalized profile as much amplitude for plotting. Note that if we had derived this from differentiating the normalized expression for flow profile then our normalization would be missing a factor of $C_{B,T}$.

Where does the shear reach an extremal value? We can see in Figure (2) that it is close to the core of the pinch. Mathematically, this will occur where the second derivative of the flow is equal to zero.

$$u''_z = \frac{\mathrm{d}u'_z}{\mathrm{d}r} = 2u_{z,0}C_{B,T}\frac{\mathrm{d}\left(r(r+C_{B,T})^{-3}\right)}{\mathrm{d}r} \tag{106}$$

$$= 2u_{z,0}C_{B,T}\left((r+C_{B,T})^{-3} - 3r(r+C_{B,T})^{-4}\right) \tag{107}$$

$$= 2u_{z,0}C_{B,T}\left(\frac{r+C_{B,T}}{(r+C_{B,T})^4} - \frac{3r}{(r+C_{B,T})^4}\right) \tag{108}$$

$$= 2u_{z,0}C_{B,T}\left(\frac{C_{B,T}-2r}{(r+C_{B,T})^4}\right) \tag{109}$$

$$= 0 \tag{110}$$

$$\implies r_{max} = \frac{C_{B,T}}{2} \tag{111}$$

The maximal shear at this point is,

$$u'_z(r_{max}) = \frac{8}{27}\frac{u_{z,0}}{C_{B,T}} \tag{112}$$

Obtaining the magnetic field for a specific instantiation of the cubic, pureflow, Bennett-Shumlak vortex requires a given plasma state that is by definition, not general. In general, the magnetic field given by Equation (69) can be normalized. We neglect the negative sign in showing this because the negative sign only describes the direction of the field, either

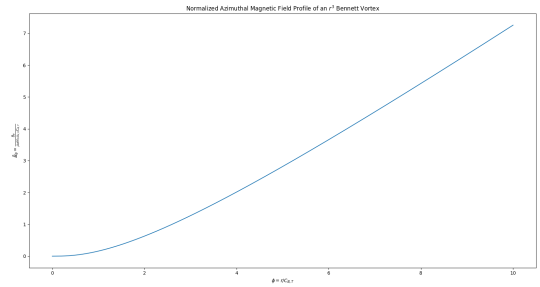

FIG. 3. Normalized magnetic field for a cubic, pureflow Bennett-Shumlak vortex over the range $\phi \in [10^{-6}, 10^1]$.

clockwise or counterclockwise about the pinch.

$$B_\theta(r) = \mu_0 e n_0 u_{z,0} \frac{f_1 + f_2 + f_3 + f_4}{2r(r + C_{B,T})} \tag{113}$$

$$= \frac{A_B C_{B,T}}{2\phi(\phi+1)} \Big(\phi^3 - 3\phi^2 \tag{114}$$

$$- 6\phi(1 - \ln(\phi+1)) + 6\ln(\phi+1)\Big) \tag{115}$$

$$A_B = \mu_0 e n_0 u_{z,0} \tag{116}$$

$$\therefore \tilde{B}_\theta(\phi) = \frac{1}{\phi(\phi+1)} \Big(\phi^3 - 3\phi^2 - 6\phi(\tag{117}$$

$$1 - \ln(\phi+1)) + 6\ln(\phi+1)\Big) \tag{118}$$

$$= B_\theta(r) \frac{2}{A_B C_{B,T}} \tag{119}$$

Let us finish our consideration of the two-fluid model which we began from by obtaining a value for the thermal power of a Bennett-Shumlak vortex in order to show that the minimum value occurs at the cubic state. The heat flux for a cubic temperature profile, and a uniform thermal conductivity reads,

$$\vec{q} = -\kappa_\perp \nabla C_T r^3 \tag{120}$$

$$= -3\kappa_\perp C_T r^2 \hat{r} \tag{121}$$

the divergence of which is,

$$\nabla \cdot \vec{q} = -3\kappa_\perp C_T \nabla \cdot r^2 \hat{r} \tag{122}$$

$$= -3\kappa_\perp C_T \frac{1}{r} \frac{\mathrm{d}(r^3)}{\mathrm{d}r} \tag{123}$$

$$= -3\kappa_\perp C_T \frac{1}{r} 3r^2 \tag{124}$$

$$= -9\kappa_\perp C_T r \tag{125}$$

$$= S \tag{126}$$

Integrating this thermal power density throughout the pinch volume we have,

$$\int_{V_p} S dV = 2\pi L \int_0^{r_p} -9\kappa_\perp C_T r^2 dr \tag{127}$$

$$= -6\pi L \kappa_\perp C_T r_p^3 \tag{128}$$

$$= -6\pi L \kappa_\perp T_p \tag{129}$$

$$= P_{th}(3) \tag{130}$$

When the temperature profile is a power-law characterized by exponent, n, we have instead,

$$\vec{q}_n = -\kappa_\perp \nabla C_T^{(n)} r^n \tag{131}$$

$$= -\kappa_\perp C_T^{(n)} n r^{n-1} \hat{r} \tag{132}$$

the divergence of which is,

$$\nabla \cdot \vec{q}_n = -n\kappa_\perp C_T^{(n)} \nabla \cdot (r^{n-1}\hat{r}) \tag{133}$$

$$= -n\kappa_\perp C_T^{(n)} \frac{1}{r} \frac{\mathrm{d}(r^n)}{\mathrm{d}r} \tag{134}$$

$$= -n^2 \kappa_\perp C_T^{(n)} r^{n-2} \tag{135}$$

so that the total thermal power becomes,

$$P_{th}(n) = \int_{V_p} S_n(r) dV = 2\pi L \int_0^{r_p} -\kappa_\perp C_T^{(n)} n^2 r^{n-1} dr \tag{136}$$

$$= -2\pi L \kappa_\perp n^2 C_T^{(n)} \int_0^{r_p} r^{n-1} dr \tag{137}$$

$$= -2\pi L \kappa_\perp n C_T^{(n)} r_p^{(n)} \tag{138}$$

$$= -2\pi L \kappa_\perp n T_p \tag{139}$$

We can see then that the cubic vortex is the shear-flow stabilized state with the minimum thermal power which satisfies the strong form of the Shumlak-Hartman criterion for the Bennett profile.

Finally, let us verify that this profile is indeed a shear-flow stabilized Z-pinch because our initial argument can be blown up if the magnetic field does not have the right structure. We must evaluate the Shumlak-Hartman criterion, Equation (5) for these profiles to determine the minimum length required for the Z-pinch equilibrium to be shear-flow stabilized. This analysis will consider the limit as the plasma radius goes to zero so that the behavior of this property in an arbitrarily small space will be elucidated.

There are four primary forms to evaluate this criterion for, namely, the cubic, n-form, and their bulk forms. However, in this paper we will only evaluate the criterion for the cubic form, and its bulk cousin. Specifically, we are evaluating the expression,

$$L > \frac{\pi}{5} (\rho \mu_0)^{-1/2} \left(\frac{\mathrm{d}u_z}{\mathrm{d}r}\right)^{-1} B_\theta \tag{140}$$

for its value as $\lim_{r\to 0}$ is taken. This expression can be simplified by considering a length that is normalized by the uniform values on the RHS, and this can include the mass density if we restrict our study to situations which admit the treatment of a uniform density. In principle, this means any plasma current density that describes a form of cubic, pureflow, Bennett-Shumlak vortex without loss of generality because a vortex with a non-uniform density can be treated as isomorphic to the case of a uniform density and the flow patterns investigated in this article. This self-similarity property also extends to vortices which are higher-order than cubic, and even ones where the nonlinearity is exchanged partially. A formal study of this exchange is outside the scope of this article.

Then, we study

$$\lim_{r\to 0} L^* > \left(\frac{\mathrm{d}u_z}{\mathrm{d}r}\right)^{-1} B_\theta \tag{141}$$

$$L^* = L/\left(\frac{\pi}{5}(\rho\mu_0)^{-1/2}\right) \tag{142}$$

The influence of a bulk flow will only impact the magnetic field form so we begin with the non-bulk cases, and first amongst them the cubic case,

$$\lim_{r\to 0} L_3^* > -\frac{en_0\mu_0}{u_{z,0}C_{B,T}} \lim_{r\to 0} \frac{(r+C_{B,T})^3}{r} \frac{f(r)}{2r(r+C_{B,T})} \tag{143}$$

$$> -\frac{en_0\mu_0}{2u_{z,0}C_{B,T}} \lim_{r\to 0} \frac{(r+C_{B,T})^2}{r^2} f(r) \tag{144}$$

Two applications of L'Hopitals rule are required to lift the singularity in the denominator. The limit we wish to evaluate then becomes,

$$\lim_{r\to 0} 2f + 4(r+C_{B,T})f' + (r+C_{B,T})^2 f'' \tag{145}$$

$$= 2f(0) + 4C_{B,T}f'(0) + C_{B,T}^2 f''(0) \tag{146}$$

The derivatives that need to be evaluated are,

$$f'(r) = -6C_{B,T}r + 3r^2 + \frac{6C_{B,T}^3}{r+C_{B,T}} + \frac{6C_{B,T}^2 r}{r+C_{B,T}} \tag{147}$$

$$-6C_{B,T}^2\left(1+\ln\left(\frac{C_{B,T}}{r+C_{B,T}}\right)\right) \tag{148}$$

$$f''(r) = -6C_{B,T} + 6r - \frac{6C_{B,T}^3}{(r+C_{B,T})^2} \tag{149}$$

$$-\frac{6C_{B,T}^2 r}{(r+C_{B,T})^2} + \frac{12C_{B,T}^2}{r+C_{B,T}} \tag{150}$$

At the origin they become, alongside $f(r)$,

$$f(0) = 0 - 0 - 6C_{B,T}^3\ln(1) - 6\left(1+\ln(1)\right) * 0 * C_{B,T}^2$$
$$= 0 \tag{151}$$

$$f'(0) = 0 + 0 + 6C_{B,T}^2 + 0 - 6C_{B,T}^2\left(1+\ln(1)\right) = 0 \tag{152}$$

$$f''(0) = -6C_{B,T} + 0 - 6C_{B,T} + 0 + 12C_{B,T} = 0 \tag{153}$$

evidently, we find,

$$\lim_{r\to 0} L_3^* > 0 \tag{154}$$

Location	mean	std	max	min
Front	0.1411	0.0697	0.3558	0.0791
Wake	0.1668	0.0406	0.2132	0.0418

TABLE I. Statistics on these analytic shear-flow stabilized Z-pinch MHD solutions to the MAST edge pedestal

This is only true if $C_{B,T} \neq 0$, a fact which is only violated non-trivially when $T_p \to \infty$ which is suggestive of an ultraviolet-type pathology that shows up here by an explosion of the minimum length required for this equilibrium to assume a shear-flow stabilized state in an arbitrarily small space towards ∞. Interestingly, when we evaluate the minimum length for an n-vortex we find the opposite. That $T_p \to \infty$ implies a minimum length of zero required for shear-flow stabilization. This is outside the scope of the paper because of the extensive mathematics involved in this derivation, but is suggestive of possible cosmological implications.

The addition of a bulk flow does not change the structure of the shear but it does change the form of the magnetic field on the RHS of the Shumlak-Hartman criterion,

$$\lim_{r\to 0} L_{3,0}^* > -\frac{en_0\mu_0}{u_{z,0}C_{B,T}} \lim_{r\to 0} \frac{(r+C_{B,T})^3}{r}\left(\frac{u_0}{2}r \pm \frac{f(r)}{2r(r+C_{B,T})}\right)$$
$$> -\frac{en_0\mu_0}{2u_{z,0}} u_0 C_{B,T}^2 \tag{155}$$

This is negative-valued, and when the pinch radius goes to zero it does as well because of the $\sim r_p^6$ scaling of the RHS.

MAST EDGE PEDESTAL PRE-ELM

Tokamaks develop an edge confinement barrier at sufficiently high heating power, and this phenomenon is associated with the development of enhanced thermodynamic gradients[6–8] and shear layers[9]. These barriers periodically relax through edge-localized modes (ELMs)[10], which are associated with the large edge pressure and current gradients that develop in the pedestal before reforming under continued plasma heating.

A parameter sweep across density and temperature with $N_{sweep} = 25^2$ was performed using the experimental data from the MAST[11] edge pedestal, and this equilibrium was solved for each instance of plasma state. This result is shown in Figure (4). The solutions shown here are the most accurate ones, and all have range-normalized relative root mean-square errors $< 10\%$,

$$RRMSE = \frac{1}{J_{max} - J_{min}} \sqrt{\frac{1}{N}\sum_{i=1}^{N}(J_z(r_i) - J_{exp}(r_i))^2} \tag{156}$$

Because the experimental data here represents a discrete set of points, (r_i, J_i), when the numerical representation of the analytic curve, $J_z(r_i)$, falls between them we linearly interpolate between the experimental data to obtain a representative value.

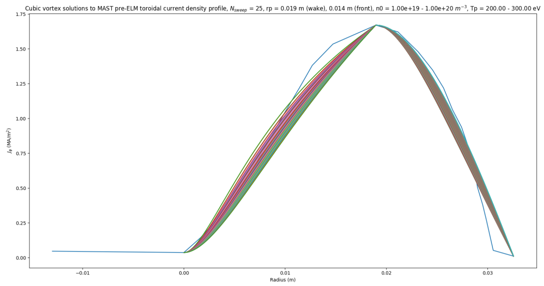

FIG. 4. Parameter sweep across the experimental conditions of the MAST edge pedestal showing solutions to this shear-flow stabilized Z-pinch equilibrium which have <10% range-normalized relative root mean-square error.

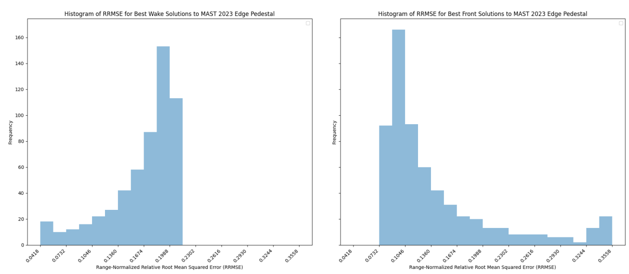

FIG. 5. Frequency of accuracy occurrence for all the best solutions obtained to the front and wake of the MAST edge pedestal by the parameter sweep through the experimental conditions.

LIMITATIONS & DISCREPANCIES

The present study was restricted to an axisymmetric, axial ansatz for the electron and ion flow in the two-fluid equations. If the pure axial nature of this were relaxed then the two-fluid drift velocity,

$$\vec{U} = \vec{u}_i - \vec{u}_e = u_i(r) - u_e(r)\hat{z} \tag{157}$$

would remain axial, and axisymmetric despite the presence of radial, and swirl components in the species velocities so long as these components are identical,

$$\vec{u}_i(r) = u_i(r)\hat{z} + u_r\hat{r} + u_\theta\hat{\theta} \tag{158}$$
$$\vec{u}_e(r) = u_e(r)\hat{z} + u_r\hat{r} + u_\theta\hat{\theta} \tag{159}$$

Even in an axisymmetric system these new components would lead to inertial forces, e.g., radial self-advection, centrifugal acceleration, radial transport of angular momentum, geometric coupling between radial and swirl motion, as well as radial transport of the axial momentum. For a more general non-axisymmetric system then components beyond these would be introduced.

DISCUSSION & CONCLUSION

The two-fluid study performed here led to the MHD momentum equation because of the ansatz,

$$\vec{u}_e(r) = u_e(r)\hat{z} \tag{160}$$
$$\vec{u}_i(r) = u_i(r)\hat{z} \tag{161}$$

as this resulted in the elimination of the convective nonlinearities and, alongside the steady density that exists when the Bennett profile is exchanged between number density and plasma velocity, the unsteady Eulerian component that makes up the rest of the inertial forces as well.

In order to link the MHD flow with the plasma current density an electron-dominated regime was studied where the ion velocity was taken to be,

$$\vec{u}_i \approx 0 \tag{162}$$

so that the MHD flow,

$$\vec{u} = \frac{m_e}{m_i + m_e}\vec{u}_e \tag{163}$$

becomes directly related to the electron velocity when the electrons are suitably relativistic, such that $m_e(\gamma) >> m_i$. This occurs without the need to introduce any Lorentz factors into the expression because they cancel from the top and bottom, leaving,

$$\vec{u} \simeq \vec{u}_e \tag{164}$$

However, the same would be true if the ions were studied from the rest frame of the electrons so that,

$$\vec{u} = \frac{m_i}{m_e + m_i}\vec{u}_i^{(R)} = \vec{u}_i + \frac{m_e}{m_i}\vec{u}_e \tag{165}$$

$$\therefore \vec{u}_i^{(R)} = \frac{m_i + m_e}{m_i}\vec{u} \tag{166}$$
$$= \vec{u}_i + \frac{m_e}{m_i}\vec{u}_e \tag{167}$$

For non-relativistic electrons and ions then the above reduces to,

$$\vec{u}_i^{(R)} \simeq \vec{u} \simeq \vec{u}_i \tag{168}$$

leaving us free to also adopt,

$$\vec{J} = en\vec{u}_i^{(R)} = en\vec{u} = en\vec{u}_i \tag{169}$$

as a means of solving the MHD momentum equation based on the MHD flow relevant to the Shumlak-Hartman shear-flow stabilization criterion. Similarly, we can also adopt the perspective of the ion rest frame so that the MHD flow becomes,

$$\vec{u} = \frac{m_e}{m_i + m_e}\vec{u}_e^{(R)} = \frac{m_i\vec{u}_i + m_e\vec{u}_e}{m_i + m_e} \tag{170}$$
$$\therefore \vec{u}_e^{(R)} = \frac{m_i}{m_e}\vec{u}_i + \vec{u}_e \tag{171}$$

which becomes,

$$\vec{u}_e^{(R)} \simeq \vec{u}_e \simeq \vec{u} \tag{172}$$

when the electrons are suitably relativistic, leaving us free to adopt the same perspective towards the plasma current density as previous. The appeal towards relativity does not alter the structure of the two-fluid equations as the lack of inertial forces in this regime will result in the $m_s(\gamma)$ terms disappearing. It also does not alter the structure of the derived profile as the introduction of relativistic length contraction, e.g.,

$$r \to \frac{r}{\gamma} \tag{173}$$

$$C_{B,T} \to \frac{C_{B,T}}{\gamma} \tag{174}$$

leads to,

$$u_z(r) \to u_{z,0}\frac{(r/\gamma)^2}{(r/\gamma + C_{B,T}/\gamma)^2} \tag{175}$$

$$= u_{z,0}\frac{\gamma^2}{\gamma^2}\frac{r^2}{(r+C_{B,T})^2} \tag{176}$$

$$= u_{z,0}\frac{r^2}{(r+C_{B,T})^2} \tag{177}$$

the same flow profile.

This lengthy re-litigation of the earlier approach to MHD supports the flexibility of this equilibrium in applying to both electron-current, and ion-current carrying regimes as well as relativistic electron regimes. There is also no problem in it describing the drift velocity either as a kind of mixed form. The only situation not considered is when the ions are relativistic, and the electrons are not. Relativistic ions would produce large electromagnetic fields that the electrons would quickly accelerate under the action of.

As far as application to the MAST edge pedestal, and edge pedestals more broadly goes, the analytic shear-flow stabilized Z-pinch hypothesis put forward here is suggested to be a strong one, but not one that claims to capture all of the physics involved in this process, or in the ELM process that occurs during periodic relaxation. For example, turbulence, viscosity, and higher-order kinetic effects are all features which have not been captured by this simple shear-flow stabilized Z-pinch profile.

The strength of this hypothesis is manifold. The shear-flow stabilization property explains the existence of the confinement barrier, the enhanced thermodynamic gradients required for its construction match observation, and the finite thermal lifetime,

$$\tau_E = \frac{1}{4}\frac{p_0}{\kappa_\perp(r_p)T_p}r_p^2 \tag{178}$$

suggests that the confinement will be relaxed when the pinch expires and the magnetic topology re-organizes. Lastly, the presence of large radial drifts due to an Ohmic electric field in the presence of a plasma resistivity, and the weak-field that exists in the core of the pinch,

$$\vec{v}_\eta = \frac{1}{q_s}\frac{q_s E_z\hat{z}\times B_\theta\hat{\theta}}{B^2} \tag{179}$$

$$= \eta_\perp\frac{J_z}{B_\theta}\hat{r} \tag{180}$$

the negative sign is cancelled in the above because the azimuthal magnetic field has a negative sign lurking in it when the electrons carry the current, and when the ions do it also doesn't matter because the large drifts are in the core of the pinch so they will still be ejected outwards.

Finally, please note that if the pinch axis were aligned with the direction of a uniform gravitational field, e.g., the Earth's, then large radial "waterfall" drifts would also buildup due to the weak-field region that exists in the core of the pinch,

$$\vec{v}_{wf} = \frac{m_j}{q_j}\frac{g_0\hat{z}\times B_\theta\hat{\theta}}{B^2} \tag{181}$$

$$= \frac{g_0}{\omega_{cj}}\hat{r} \tag{182}$$

which mathematically supports the hypothesis that these drifts could be used to eject ionized pollutants from a stream because of the large amplitude collisionless drifts that exist in the core of the pinch. The collisional picture affects this,

$$u_r = g_0\frac{\omega_{cj}}{\nu^2+\omega_{cj}^2} = v_{wf}^{(coll.)}(r) \tag{183}$$

but analyzing this any further is beyond the scope of this article which intends just to present this analytic shear-flow stabilized Z-pinch equilibrium, and its ability to solve the edge pedestal of MAST, as well as explain the emergence of the edge pedestal phenomenon.

One enduring question is whether there are other shear-flow stabilized Z-pinch equilibria that can be derived beyond this. The tendency of the author is to say "yes" just so as to not potentially leave the subject prematurely unclosed. However, the question is "how"? How would another be constructed that did not fall into this family of equilibria, and which possessed the same or more significance? The author does not know.

CODE

The code solving the MAST parameter sweep is here, and the code creating the histogram is here.

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DATA AVAILABILITY STATEMENT

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