On the Collisional Ordering of a Shear-Flow Stabilized Cubic, Pureflow Bennett-Shumlak-Hartman Vortex

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(Dated: 25 June 2026)

This brief note shows that the ion fluid momentum reduces to the Ideal Ohm's Law when the standard ordering $\omega_{pe} >> \bar{\nu}_{ei}$ holds for the non-relativistic Braginskii Coulomb collision frequency.

The ion momentum equation reads,

$$m_i n_i \left(\frac{\partial}{\partial t} + \vec{u}_i \cdot \nabla\right)\vec{u}_i = en_i(\vec{E} + \vec{u}_e \times \vec{B}) - \nabla p_i - m_e n_e \bar{\nu}_{ei}(\vec{u}_i - \vec{u}_e) \tag{1}$$

where the equilibrium-averaged Braginskii electron-ion collision frequency,

$$\bar{\nu}_{ei} = (2.9 * 10^{-6})\frac{n_e \ln(\Lambda)}{T_e^{3/2}} \tag{2}$$

Unlike in the electron case no ad hoc relativistic modifications are required to obtain the Ideal Ohm's Law, only the standard plasma ordering,

$$\omega_{pe} >> \bar{\nu}_{ei} \tag{3}$$

and the Z-pinch ansatz,

$$\vec{u}_i = u_i(r)\hat{z} \tag{4}$$

The electron plasma frequency is,

$$\omega_{pe}^2 = \frac{n_e e^2}{m_e \varepsilon_0} \tag{5}$$

so that with quasineutrality,

$$n_e = n_i \tag{6}$$

we have for the ion momentum equation,

$$m_i n_e \left(\frac{\partial}{\partial t} + \vec{u}_i \cdot \nabla\right)\vec{u}_i = en_e(\vec{E} + \vec{u}_i \times \vec{B}) - \nabla p_i - m_e n_e \bar{\nu}_{ei}(\vec{u}_i - \vec{u}_e) \tag{7}$$

which becomes,

$$\therefore m_i \frac{m_e \varepsilon_0}{e^2}\omega_{pe}^2 \left(\frac{\mathrm{d}}{\mathrm{d}t} + \vec{u}_i \cdot \nabla\right)\vec{u}_i \tag{8}$$

$$= \frac{m_e \varepsilon_0}{e}\omega_{pe}^2(\vec{E} + \vec{u}_i \times \vec{B}) - \nabla p_i - \frac{m_e^2 \varepsilon_0}{e^2}\omega_{pe}^2 \bar{\nu}_{ei}(\vec{u}_i - \vec{u}_e) \tag{9}$$

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Dividing both sides by $\frac{1}{\omega_{pe}^3}$, we obtain,

$$\therefore m_i \frac{m_e \varepsilon_0}{e^2}\frac{1}{\omega_{pe}} \left(\frac{\mathrm{d}}{\mathrm{d}t} + \vec{u}_i \cdot \nabla\right)\vec{u}_i \tag{10}$$

$$= \frac{m_e \varepsilon_0}{e}\frac{1}{\omega_{pe}}(\vec{E} + \vec{u}_i \times \vec{B}) - \frac{1}{\omega_{pe}^3}\nabla p_i - \frac{m_e^2 \varepsilon_0}{e^2}\frac{\bar{\nu}_{ei}}{\omega_{pe}}(\vec{u}_i - \vec{u}_e) \tag{11}$$

the collisional term then disappears because of the aforementioned non-relativistic ordering. Similarly, the inertial forces on the LHS will as well because of the Z-pinch ansatz. This leaves,

$$\frac{m_e \varepsilon_0}{e}\frac{1}{\omega_{pe}}(\vec{E} + \vec{u}_i \times \vec{B}) - \frac{1}{\omega_{pe}^3}\nabla p_i = 0 \tag{12}$$

$$\rightarrow \frac{m_e \varepsilon_0}{e}\frac{1}{\omega_{pe}}(\vec{E} + \vec{u}_i \times \vec{B}) = \frac{1}{\omega_{pe}^3}\nabla p_i \tag{13}$$

Let us multiply through by $\omega_{pe}^2 \bar{\nu}_{ei}$, obtaining,

$$\frac{m_e \varepsilon_0}{e}\omega_{pe}\bar{\nu}_{ei}(\vec{E} + \vec{u}_i \times \vec{B}) = \frac{\bar{\nu}_{ei}}{\omega_{pe}}\nabla p_i \tag{14}$$

so that for sufficiently small pressure gradients we can eliminate the term on the RHS asympototically, yielding,

$$\frac{m_e \varepsilon_0}{e}\omega_{pe}\bar{\nu}_{ei}(\vec{E} + \vec{u}_i \times \vec{B}) = 0 \tag{15}$$

which requires,

$$\vec{E} = -\vec{u}_i \times \vec{B} \tag{16}$$

linking the MHD flow with the two-fluid drift velocity then requires stationary electrons because for non-relativistic plasmas the MHD flow is naturally very close to the ion flow,

$$\vec{u} \simeq \vec{u}_i \tag{17}$$

and with stationary electrons the same is true for the two-fluid drift,

$$\vec{U} = \vec{u}_i - \vec{u}_e \simeq \vec{u}_i \tag{18}$$

the physical picture behind this is that when the electrons are oscillating extremely fast about a central point, then from the perspective of the ions their motion is so fast that in an average sense do not appear to move from this central point.