Bennett-Shumlak Vortices are MHD: Notes about the Two-Fluid Theory and Electron-Dominated MHD

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Two-Fluid Theory

From the First Edition of Jeffrey Friedberg's *Plasma Physics and Fusion Energy* published August 11th, 2008, Equation (11.2) we have the two-fluid model of Magnetohydrodynamics, with the electron inertia retained,

$$\frac{\partial n_e}{\partial t} + \nabla \cdot (n_e \vec{u}_e) = 0 \tag{1}$$

$$\frac{\partial n_i}{\partial t} + \nabla \cdot (n_i \vec{u}_i) = 0 \tag{2}$$

$$m_e n_e \left(\frac{\partial}{\partial t} + \vec{u}_e \cdot \nabla \right) \vec{u}_e = -e n_e (\vec{E} + \vec{u}_e \times \vec{B}) - \nabla p_e - m_e n_e \bar{\nu}_{ei} (\vec{u}_e - \vec{u}_i) \tag{3}$$

$$m_i n_i \left(\frac{\partial}{\partial t} + \vec{u}_i \cdot \nabla \right) \vec{u}_i = e n_i (\vec{E} + \vec{u}_i \times \vec{B}) - \nabla p_i - m_e n_e \bar{\nu}_{ei} (\vec{u}_i - \vec{u}_e) \tag{4}$$

$$\frac{3}{2} n_e (\frac{\partial}{\partial t} + \vec{u}_e \cdot \nabla) T_e + p_e \nabla \cdot \vec{u}_e + \nabla \cdot \vec{q}_e = S_e \tag{5}$$

$$\frac{3}{2} n_i (\frac{\partial}{\partial t} + \vec{u}_i \cdot \nabla) T_i + p_i \nabla \cdot \vec{u}_i + \nabla \cdot \vec{q}_i = S_i \tag{6}$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{7}$$

$$\nabla \times \vec{B} = \mu_0 e (n_i \vec{u}_i - n_e \vec{u}_e) \tag{8}$$

$$n_i - n_e = 0 \tag{9}$$

$$\nabla \cdot \vec{B} = 0 \tag{10}$$

With quasineutrality, a hydrogen plasma, and Ampere's Law we have,

$$\nabla \times \vec{B} = \mu_0 e n (\vec{u}_i - \vec{u}_e) \tag{11}$$

$$= \mu_0 e n \vec{u} \tag{12}$$

as one way of looking at it. It is absolutely critical to note at this junction that regardless of our interpretation of the plasma flow,

$$\vec{u} = \vec{u}_i - \vec{u}_e \tag{13}$$

when this flow takes the form of a cubic, pureflow Bennett vortex, which is a kind of shear-flow stabilized Z-pinch equilibrium,

$$\begin{aligned} \vec{u} &= u_z(r)\hat{z} && (14) \\ &= u_{z,0}\frac{r^2}{(r+C_{B,T})^2} && (15) \end{aligned}$$

then we have an analytic solution for the magnetic field from the above,

$$\begin{aligned} \vec{B} &= B_\theta(r)\hat{\theta} && (16) \\ &= \frac{\mu_0 e n_0 u_{z,0}}{2r(r+C_{B,T})} f(r, C_{B,T})\hat{\theta} && (17) \end{aligned}$$

where,

$$\begin{aligned} f(r, C_{B,T}) &= f_1 + f_2 + f_3 + f_4 && (18) \\ f_1(r) &= r^3 && (19) \\ f_2(r, C_{B,T}) &= -3r^2 C_{B,T} && (20) \\ f_3(r, C_{B,T}) &= -6rC_{B,T}^2(1 + \ln\left(\frac{C_{B,T}}{r+C_{B,T}}\right)) && (21) \\ f_4(r, C_{B,T}) &= -6C_{B,T}^3 \ln\left(\frac{C_{B,T}}{r+C_{B,T}}\right) && (22) \end{aligned}$$

This satisfies the eponymous "Sad Law" stating there are no magnetic monopoles in the framework of classical electromagnetism, or more strictly, that the magnetic field is divergenceless,

$$\begin{aligned} \nabla \cdot \vec{B} &= \nabla \cdot B_\theta(r)\hat{\theta} && (23) \\ &= \frac{1}{r}\frac{\partial(rB_r)}{\partial r} + \frac{1}{r}\frac{\partial B_\theta}{\partial \theta} + \frac{\partial B_z}{\partial z} && (24) \\ &= 0 && (25) \end{aligned}$$

Faraday's Law is automatically satisfied for the above because the magnetic field is steady, and the electric field for the axisymmetric shear-flow stabilized Z-pinch described by this combination of axial flow and azimuthal magnetic field is itself axisymmetric so that,

$$\begin{aligned} \nabla \times \vec{E} &= \left(\frac{1}{r}\frac{\partial E_z}{\partial \theta} - \frac{\partial E_\theta}{\partial z}\right)\hat{r} + \left(\frac{\partial E_r}{\partial z} - \frac{\partial E_z}{\partial r}\right)\hat{\theta} + \left(\frac{1}{r}\frac{\partial (rE_\theta)}{\partial r} - \frac{1}{r}\frac{\partial E_r}{\partial \theta}\right)\hat{z} && (26) \\ &= 0 && (27) \end{aligned}$$

To reduce the two-fluid momentum to the ideal case we consider the ansatz,

$$\vec{u}_e = u_e(r)\hat{z} \tag{28}$$

$$\vec{u}_i = u_i(r)\hat{z} \tag{29}$$

which is a valid ansatz to consider. It is possible for axisymmetric radial and swirling terms to exist in the above so long as they are equal. However, this will lead to additional physics which is outside the scope of this article.

Then, we have from the two-fluid momentum equations for a hydrogen plasma written above,

$$en(\vec{u}_i - \vec{u}_e) \times \vec{B} = \vec{J} \times \vec{B} = \nabla p \tag{30}$$

because the convective nonlinearities will disappear for this ansatz,

$$\hat{r} \cdot (\vec{u} \cdot \nabla)\vec{u} = u_r \frac{\partial u_r}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta} + u_z \frac{\partial u_r}{\partial z} - \frac{u_\theta^2}{r} \tag{31}$$

$$\hat{\theta} \cdot (\vec{u} \cdot \nabla)\vec{u} = u_r \frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_\theta}{\partial \theta} + u_z \frac{\partial u_\theta}{\partial z} + \frac{u_\theta u_r}{r} \tag{32}$$

$$\hat{z} \cdot (\vec{u} \cdot \nabla)\vec{u} = u_r \frac{\partial u_z}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_z}{\partial \theta} + u_z \frac{\partial u_z}{\partial z} \tag{33}$$

and because the friction forces are equal and opposite so that they will cancel which does not require the species to be collisionless. This leaves the Ideal MHD force balance and for the axisymmetric shear-flow stabilized Z-pinch here leads to,

$$\frac{\mathrm{d}p}{\mathrm{d}r} = -J_z B_\theta \tag{34}$$

where

$$p = p_e + p_i \tag{35}$$

Before continuing to the two-fluid energy let us also note that for this uniform density, steady axial flow that we have continuity satisfied,

$$\frac{\partial n_i - n_e}{\partial t} + n_i \nabla \cdot \vec{u}_i - n_e \nabla \cdot \vec{u}_e = 0 \tag{36}$$

The time derivative in the above evidently disappears for a uniform density, or a quasineutral one, and the axisymmetric axial flow described here is divergenceless,

$$\nabla \cdot (\vec{u}_i - \vec{u}_e) = \nabla \cdot \vec{u} = 0 \tag{37}$$

The only aspect left of these equations to consider here are the energy equations. Let us address them. The exact form of the electron and ion velocities here does not matter so long as they are both axisymmetric and purely axial. In that case we have much cancellation occur because,

$$\frac{3}{2} n \left(\frac{\partial}{\partial t} + u_e(r)\hat{z} \cdot \nabla \right) T_e(r) + p_e \nabla \cdot u_e(r)\hat{z} = 0 \tag{38}$$

and similarly for the ions. This leaves,

$$\nabla \cdot \vec{q}_e = S_e \tag{39}$$
$$\nabla \cdot \vec{q}_i = S_i \tag{40}$$

which can be added together to obtain,

$$\nabla \cdot \vec{q} = S \tag{41}$$

Based on a form for the heat flux of,

$$\begin{aligned}
\vec{q} &= -\kappa_\perp \nabla T(r) && (42)\\
&= -\kappa_\perp \nabla C_T^{(3)} r^3 && (43)\\
&= -\kappa_\perp \nabla \frac{T_p}{r_p^3} r^3 && (44)\\
&= -\kappa_\perp \frac{T_p}{r_p^3} \nabla r^3 && (45)\\
&= -\kappa_\perp \frac{T_p}{r_p^3} 3r^2 \hat{r} && (46)
\end{aligned}$$

we then have for the source of thermal energy density on the RHS,

$$\begin{aligned}
\nabla \cdot \vec{q} &= -\kappa_\perp \frac{T_p}{r_p^3} \nabla \cdot (3r^2 \hat{r}) && (47)\\
&= -\kappa_\perp \frac{T_p}{r_p^3} \frac{1}{r} \frac{\mathrm{d}(3r^3)}{\mathrm{d}r} && (48)\\
&= -\kappa_\perp \frac{9T_p}{r_p^3} r && (49)\\
&= S(r) && (50)
\end{aligned}$$

Integrating this throughout the pinch volume we have,

$$\begin{aligned}
\int_V S(r) dV &= -9\kappa_\perp \frac{T_p}{r_p^3} \int_0^{r_p} \int_0^{2\pi} \int_0^L r^2 dr && (51)\\
&= -18\pi \kappa_\perp T_p L \frac{1}{r_p^3} \int_0^{r_p} r^2 dr && (52)\\
&= -6\pi \kappa_\perp T_p L && (53)
\end{aligned}$$

which is exactly the result obtained from the cubic pureflow vortex theory.

From studying the Two-fluid theory we have found that the shear-flow stabilized cubic pureflow vortex Z-pinch profile satisfies the entire set of equations and gives the same results as previously found in the MHD study that was performed.

Electron-Dominated Ideal MHD

The typical way to study the ideal behavior of a magnetohydrodynamic fluid involves neglecting electron inertia so that the ions are treated as carrying the bulk of the momentum. However, that is only true when the ions have an appreciable momentum compared to the electrons. If the ions are immobile from the perspective of the electrons, or just have their inertia dominated, e.g., from relativistic electrons, then the MHD momentum density,

$$\rho\vec{u} = mn\vec{u} = \rho_e\vec{u}_e + \rho_i\vec{u}_i \tag{54}$$

$$= m_e n_e\vec{u}_e + m_i n_i\vec{u}_i \tag{55}$$

obviously becomes,

$$\rho\vec{u} = \rho_e\vec{u}_e \tag{56}$$

when the ions are treated as being immobile or having their momentum dominated by relativistic electrons. Note that this is physical due to the electron response to external electric fields occuring on a much shorter timescale than the ion response due to their lower inertia at non-relativistic speeds. At relativistic speeds there will be a point where the ion inertia is the lower of the two, and then they will respond quicker. However, until this point is reached the electrons will respond to an applied electric field first, and the above is a valid perspective of looking at MHD provided that they are moving sufficiently faster than the ions, but not so fast that the ions begin responding first to the electric fields instead.

However, the ions still retain a sense of inertia that shows up in this picture,

$$\rho = m_i n_i + m_e n_e = (m_i + m_e)n \tag{57}$$

so that the MHD velocity in this framework is related to the electron velocity,

$$\vec{u} = \frac{m_e}{m_e + m_i}\vec{u}_e \tag{58}$$